·<u>DEF. 3.11</u>: Let M be a smooth mnfd. The <u>tangent bundle</u> of M is denoted by TM and is defined as the disjoint union of the tangent spaces at all pts of M:

$$TM = \bigsqcup_{p \in M} T_pM.$$

We usually write an element of this disjoint union as an ordered pair (p,v) with $p \in M$ and $v \in T_pM$ (we sometimes write v_p for (p,v)). The tangent bundle comes equipped with a natural projection map $\pi : TM \longrightarrow M$, which sends each vector in T_pM to the point p at which it is tangent: $(p,v) \mapsto p$.

For example, when $M = \mathbb{R}^n$, using PROP. 3.3 we see that

$$T(\mathbb{R}^n) = \bigsqcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n \cong \bigsqcup_{p \in \mathbb{R}^n} \mathbb{R}^n_p = \bigsqcup_{p \in \mathbb{R}^n} \{p\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

An element of this Cartesian product can be thought of as representing either the geometric tangent vector v_p or the derivation $D_v|_p$ defined in PROP. 3.3. In general, however, the tangent bundle of a smooth mnfd cannot be identified in a natural way with a Cartesian product, because there is no canonical way to identify tangent spaces at distinct pts with each other.

The next proposition shows that the tangent bundle of a smooth mnfd can be considered as a smooth mnfd in its own right. For its proof we need LEM 1.9 (smooth mnfd chart lemma).

·<u>PROP. 3.12</u>: For any smooth n-mnfd M, the tangent bundle TM has a natural topology and smooth structure that make it into a smooth $(2n)$-mnfd. W.r.t. this structure, the projection $\pi : TM \to M$ is smooth.

<u>PROOF</u>: We begin by defining the maps that will become our smooth charts. Given any smooth chart (U, φ) for M, observe that $\pi^{-1}(U)$ is the set of all tangent vectors to M at all pts of U. Denote by $(x^1, \ldots, x^n)$ the coordinate fncts of φ, and define a map

$$\tilde{\varphi} : \pi^{-1}(U) \longrightarrow \mathbb{R}^{2n}$$

$$\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\Big|_p \right) = \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n \right) \qquad (*_6)$$

Its image is the set $\varphi(U) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. It is a bijection onto its image, because its inverse can be explicitly written as

$$\tilde{\varphi}^{-1}(x^1, \ldots, x^n, v^1, \ldots, v^n) = v^i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)} .$$

Now, suppose we are given two smooth charts (U, φ) and (V, ψ) for M, and consider the corresponding "charts" $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$ for TM. The sets

$$\tilde{\varphi}\left(\pi^{-1}(U) \cap \pi^{-1}(V) \right) = \varphi(U \cap V) \times \mathbb{R}^n$$

and

$$\tilde{\psi}\left(\pi^{-1}(U) \cap \pi^{-1}(V) \right) = \psi(U \cap V) \times \mathbb{R}^n$$

are open in $\mathbb{R}^{2n}$, and the transition map

$$\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n$$

can be written explicitly as

$$\tilde{\varphi}\circ\tilde{\varphi}^{-1}(x^1,\dots,x^n,v^1,\dots,v^n) = \tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)}\right)$$

$$\overset{(*_5)}{=} \tilde{\varphi}\left(\left(v^i \frac{\partial \tilde{x}^j}{\partial x^i}\right)\frac{\partial}{\partial \tilde{x}^j}\Big|_{\varphi^{-1}(x)}\right)$$

$$= \left(\tilde{x}^1(x),\dots,\tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^i}v^i,\dots,\frac{\partial \tilde{x}^n}{\partial x^i}v^i\right),$$

which is clearly smooth.

Choosing a countable cover $\{U_i\}$ of M by smooth coordinate domains, we obtain a countable cover of TM by coordinate domains $\{\pi^{-1}(U_i)\}$ satisfying conditions (i)-(iv) of LEM 1.9. To check the Hausdorff condition (v), just note that any two pts in the same fiber of π lie in one chart, while if (p,v) and (q,w) lie in different fibers, there exist disjoint smooth coordinate domains U and V for M s.t. $p\in U$ and $q\in V$, and then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint coordinate neighborhoods containing (p,v) and (q,w), respectively. This completes the proof of the first part of the statement.

Finally, to check that $\pi: TM \to M$ is smooth, note that w.r.t. charts (U,φ) for M and $(\pi^{-1}(U),\tilde{\varphi})$ for TM, its coordinate representation $\varphi\circ\pi\circ\tilde{\varphi}^{-1}$ is $\pi(x,v) = x$. ■

The coordinates (x^i, v^i) given by $(*_6)$ are called <u>natural coordinates on TM</u>.

⇝ global differential : ES5

⇝ basic properties of global differential : ES5

· PROP. 3.13: If M is a smooth n-mnfd which can be covered by a single smooth chart (U,φ), then its tangent bundle TM is diffeomorphic to $\varphi(U)\times\mathbb{R}^n$.

PROOF: If (U,φ) is a global smooth chart for M, then φ is, in particular, a diffeomorphism from $U=M$ to an open subset $\hat{U}\subseteq\mathbb{R}^n$. The proof of PROP. 3.12 showed that the natural coordinate chart $\tilde{\varphi}$ is a bijection from TM to $\hat{U}\times\mathbb{R}^n$, and the smooth structure on TM is defined essentially by declaring $\tilde{\varphi}$ to be a diffeomorphism. ∎

COMMENT: In general, the tangent bundle is not globally diffeomorphic (or even homeomorphic) to a product of the mnfd with $\mathbb{R}^n$.

CH. 4 : MAPS OF CONSTANT RANK

· DEF. 4.1 : Let M and N be smooth mnfds. Given a smooth map $F: M \to N$ and a pt $p \in M$, the rank of F at p is defined to be the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$; it is the rank of the Jacobian matrix of F in any smooth chart, or the dimension of the image $Im(dF_p) \subseteq T_{F(p)}N$. If F has the same rank r at every pt, we say that it has constant rank, and we write $rk F = r$.

Recall that the rank of F at each pt is bounded above by $\min\{\dim M, \dim N\}$. If the rank of dF_p is equal to this upper bound, then we say that F has full rank at p. If F has full rank everywhere, we say that F has full rank.

· DEF. 4.2 : A smooth map $F: M \to N$ is called

(a) a smooth immersion if its differential is injective at each pt (or equivalently, $rk F = \dim M$),

(b) a smooth submersion if its differential is surjective at each pt (or equivalently, if $rk F = \dim N$), and

(c) a smooth embedding if it is a smooth immersion that is also a topological embedding, i.e., a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology.

COMMENT:

1) A smooth embedding is a map that is both a topological embedding and a smooth immersion, not just a topological

embedding that happens to be smooth; see EX. 4.5(1).

2) We will see that smooth immersions and submersions behave locally like injective and surjective linear maps, resp.

· LEM 4.3: Let $F: M \to N$ be a smooth map. If dF_p is injective (resp. surjective) for some $p \in M$, then p has a neighborhood U s.t. $F|_U$ is an immersion (resp. submersion).

PROOF: If we choose any smooth coordinates for M near p and for N near $F(p)$, either hypothesis means that the Jacobian matrix of F in coordinates has full rank at $p \in M$. By ES2E4 we know that the set of $n \times m$ matrices of full rank is an open subset of $M(n \times m, \mathbb{R})$ (where $m = \dim M$ and $n = \dim N$), so by continuity, the Jacobian of F (in coordinates) has full rank in some neighborhood of $p \in M$. ∎

· EXAMPLE 4.4:

1) If $\gamma: J \to M$ is a smooth curve in a smooth mnfd M, then γ is an immersion iff $\gamma'(t) \neq 0$ for all $t \in J$; see ES4.

2) If M is a smooth mnfd and its tangent bundle TM is given the smooth mnfd structure described in PROP. 3.12, then the projection $\pi: TM \to M$ is a smooth submersion. Indeed, we saw that w.r.t. any smooth local coordinates (x^i) on an open subset $U \subseteq M$ and the corresponding natural coordinates (x^i, v^i) on $\pi^{-1}(U) \subseteq TM$, the coordinate representation of π

is $\hat{\pi}(x,v) = x$, and thus $J_{\hat{\pi}} = \begin{pmatrix} Id_{\dim M} & 0 \\ 0 & 0 \end{pmatrix}$.

3) If M is a smooth mnfd and $U \subseteq M$ is an open subset, then the inclusion map $U \hookrightarrow M$ is a smooth embedding.

$\rightsquigarrow$ for further examples, see ES6 and ES7.

To understand more fully what it means for a map to be a smooth embedding, it is useful to bear in mind some examples of injective smooth maps that are not smooth embeddings. The next three examples illustrate three rather different ways in which this can happen.

· EXAMPLE 4.5:

1) The map

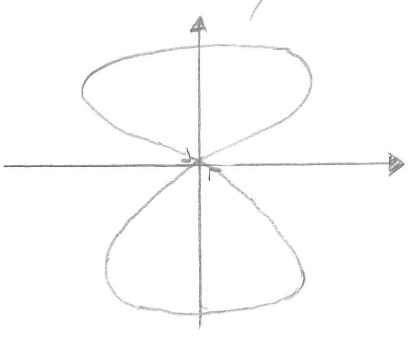
$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2, \ t \longmapsto (t^3, 0)$$

is a smooth map and a topological embedding, but it is not a smooth embedding, because $\gamma'(0) = 0$.

2) (lemniscate): Consider the curve

$$\beta : (-\pi,\pi) \longrightarrow \mathbb{R}^2, \ t \longmapsto (\sin 2t, \sin t).$$

Its image is a set that looks like a figure-eight in the plane (it is the locus of pts $(x,y) \in \mathbb{R}^2$ s.t. $x^2 = 4y^2(1-y^2)$, as one can easily check). We compute



· $\|\beta(t)\|^2 = \sin^2(t)(2\cos^2 t + 1)$, $t \in (-\pi,\pi)$

· $\|\beta'(t)\|^2 = \|(2\cos 2t, \cos t)\|^2$

$= 4\cos^2 2t + \cos^2 t$, $t \in (-\pi,\pi)$

and hence B is an injective smooth immersion, because $B(t_1)=B(t_2) \Rightarrow t_1=t_2$ and $B'(t) \neq 0$, $\forall t \in (-\pi,\pi)$. However, B is not a topological embedding, because its image is compact in the subspace topology, while its domain is not.

3) Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^2$ denote the torus, and let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The map

$$\gamma : \mathbb{R} \to \mathbb{T}^2, \; t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$$

is a smooth immersion, because $\gamma'(t)$ never vanishes. It is also injective, because

$$\gamma(t_1) = \gamma(t_2) \Rightarrow t_1 - t_2, \alpha t_1 - \alpha t_2 \in \mathbb{Z} \Rightarrow t_1 = t_2 .$$

However, γ is not a topological embedding. Indeed, using Dirichlet's approximation theorem [Lee, Lemma 4.21], one can show that $\gamma(0)$ is a limit point of $\gamma(\mathbb{Z}) = \{\gamma(n) \mid n \in \mathbb{Z}\}$, while $\mathbb{Z}$ has no limit point in $\mathbb{R}$. (It can also be shown that $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.)

The following proposition gives a few simple sufficient criteria for an injective immersion to be an embedding.

·PROP. 4.6: Let $F: M \to N$ be an injective smooth immersion. If any of the following holds, then F is a smooth embedding.

(a) F is an open or a closed map.

(b) F is a proper map (i.e., for every compact subset $K \subseteq N$, the preimage $F^{-1}(K) \subseteq M$ is compact).

(c) M is compact.

(d) $\dim M = \dim N$.

PROOF :

- Claim 1 : Let $F: X \to Y$ be a cont. map between top. sp. that is either open or closed. If F is injective, then it is a top. embedding.

- Proof : Assume that F is open and injective. Then $F: X \to F(X)$ is bijective, so $F^{-1}: F(X) \to X$ exists. If $U \subseteq X$ is open, then $(F^{-1})^{-1}(U) = F(U)$ is open in Y by hypothesis, and therefore is also open in $F(X)$ by dfn of the subspace top. on $F(X)$. Hence, F^{-1} is cont, so that F is a top. embedding.

The proof of the assertion is similar when F is closed and injective.

- Claim 2 (Closed map lemma) : Let X be a compact space, Y be a Hausdorff space, and $F: X \to Y$ be a cont. map. Then F is a closed map.

- Proof : Let $K \subseteq X$ be a closed subset. Since X is compact, K is also compact, and since F is cont, $F(K)$ is also compact. Since Y is Hausdorff, $F(K) \subseteq Y$ is a closed subset. Thus, F is a closed map.

- Claim 3 : Let X be a top. sp. and let Y be a locally compact Hausdorff space. Then every proper cont. map $F: X \to Y$ is closed.

- **Proof**: Let $K \subseteq X$ be a closed subset. T.s.t. $F(K) \subseteq Y$ is closed, w.w.s.t. its complement is open. Since $Y \setminus F(K)$ ($\overset{open}{\subseteq} Y$) is locally compact, y has an open neighborhood V with compact closure in $Y \setminus F(K)$, and since F is proper, $F^{-1}(\overline{V})$ is compact. Set $E := K \cap F^{-1}(\overline{V})$ and note that E is a compact set. Since F is cont, $F(E)$ is also compact, and since Y is Hausdorff, $F(E)$ is a closed subset of Y. Set $U := V \setminus F(E) = V \cap (Y \setminus F(E))$ and observe that U is open neighborhood of y, which is disjoint from $F(K)$. Hence, $Y \setminus F(K)$ is open, which implies that $F(K)$ is closed.

(a) By Claim 1, F is a top. embedding, and by assumption F is a smooth immersion, so it is a smooth embedding.

(b) By assumption and by Claim 3, F is a closed map, so it is a smooth embedding by (a).

(c) By assumption and by Claim 2, F is a closed map, so it is a smooth embedding by (a).

(d) By assumption and by ES6E5(b), F is a local diffeomorphism (see ES6), and thus an open map by ES6E4(c), so it is a smooth embedding by (a).

L6 ∎

- **THM** (Inverse fnct thm for mnfds): Let $F: M \to N$ be a smooth map. If $p \in M$ is a pt s.t. the differential dF_p of F at p is invertible, then there exist connected neighborhood U_0 of p in M and V_0 of $F(p)$ in N s.t. $F|_{U_0}: U_0 \to V_0$ is a diffeomorphism.