Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 7

Exercise 1:

(a) Let N and $M_1, \dots, M_k$ be smooth manifolds, where $k \geq 2$, and let $F_i \colon N \to M_i$ be smooth maps, where $1 \leq i \leq k$. Show that the map

$$G \colon N \to \prod_{i=1}^{k} M_i, \; x \mapsto \big(F_1(x), \dots, F_k(x)\big)$$

is smooth and that its differential at any point $p \in N$ is of the form

$$\big(dG_p\big)(v) = \big(d(F_1)_p(v), \dots, d(F_k)_p(v)\big), \; v \in T_pN.$$

(b) Let M be a smooth manifold. Show that there exists a smooth map $f \colon M \to [0, +\infty)$ that is proper.

[Hint: Use a function of the form $f = \sum_{i=1}^{+\infty} c_i \psi_i$, where $(\psi_i)_{i=1}^{+\infty}$ is a partition of unity and the c_i's are real numbers.]

(c) Let $F \colon M \to N$ be an injective smooth immersion between smooth manifolds. Show that there exists a smooth embedding $G \colon M \to N \times \mathbb{R}$.

[Hint: Use parts (a) and (b).]

Exercise 2 (to be submitted by Friday, 10.11.2023, 20:00):

(a) Show that the inclusion map $\iota \colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding.

(b) Consider the map

$$F \colon \mathbb{R} \to \mathbb{R}^2, \; t \mapsto (2 + \tanh t) \cdot (\cos t, \sin t).$$

(i) Show that F is an injective smooth immersion.

(ii) Show that F is a smooth embedding.

Exercise 3 (*Local embedding theorem*)**:**
Let $F\colon M \to N$ be a smooth map between smooth manifolds. Show that F is a smooth immersion if and only if every point in M has a neighborhood U such that $F|_U\colon U \to N$ is a smooth embedding.

[Hint: Use the rank theorem and the closed map lemma.]

Exercise 4:
Let M and N be smooth manifolds, and let $\pi\colon M \to N$ be a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of *Theorem 4.12*; in other words, assuming that $\widetilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P, a map $F\colon \widetilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\widetilde{N}$.

Exercise 5:
Consider the map
$$\pi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto xy.$$
Show that π is surjective and smooth, and that for each smooth manifold P, a map $F\colon \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion. (Therefore, the converse of *Theorem 4.12* is false.)

Exercise 6 (*Uniqueness of smooth quotients*)**:**
Let $\pi_1\colon M \to N_1$ and $\pi_2\colon M \to N_2$ be surjective smooth submersions that are constant on each other's fibers. Show that there exists a unique diffeomorphism $F\colon N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$: