Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 6 – Solutions

Exercise 1:

(a) Prove the following assertions:

(i) A composition of smooth submersions is a smooth submersion.

(ii) A composition of smooth immersions is a smooth immersion.

(iii) A composition of smooth embeddings is a smooth embedding.

(b) Show by means of a counterexample that a composition of smooth maps of constant rank need not have constant rank.

Solution:

(a) First, we show (i). Let $F\colon M \to N$ and $G\colon N \to P$ be smooth submersions and fix $p \in M$. Then the composite map $G \circ F\colon M \to P$ is smooth by part (e) of *Exercise* 3, *Sheet* 3, and by part (d) of *Exercise* 1, *Sheet* 4 its differential at p is the linear map

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \to T_{(G\circ F)(p)}P,$$

which is surjective, since both linear maps

$$dF_p\colon T_pM \to T_{F(p)}N \quad \text{and} \quad dG_{F(p)}\colon T_{F(p)} \to T_{(G\circ F)(p)}$$

are surjective by assumption. Since $p \in M$ was arbitrary, we conclude that $G \circ F$ is a smooth submersion.

Next, to prove (ii), we argue exactly as in (i), except that the word "surjective" is replaced by the word "injective".

Finally, we show (iii). Let $F\colon M \to N$ and $G\colon N \to P$ be smooth embeddings. By (ii) we know that the composite map $G \circ F\colon M \to P$ is a smooth immersion, so it remains to show that $G \circ F$ is a homeomorphism onto its image $(G \circ F)(M) \subseteq P$ in the subspace topology. To this end, note that F is a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology, and that G is a homeomorphism onto its image $G(N) \subseteq P$ in the subspace topology, so the restriction $G|_{F(M)}\colon F(M) \to G\big(F(M)\big)$ is also a homeomorphism. Therefore, the composite map $G \circ F$ is a homeomorphism onto its image $(G \circ F)(M) \subseteq P$ in the subspace topology, as required. In conclusion, $G \circ F$ is a smooth embedding.

(b) Consider the maps

$$\gamma\colon (0, 2\pi) \to \mathbb{R}^2,\ t \mapsto (\cos t, \sin t)$$

and

$$\pi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto y.$$

By *Exercise* 2(a), π is a surjective smooth submersion. Moreover, we have

$$\|\gamma(t)\| = 1 \quad \text{and} \quad \|\gamma'(t)\| = \|(-\sin t, \cos t)\| = 1 \ \text{ for all } t \in (0, 2\pi),$$

so γ is an injective smooth immersion; see *Example 4.4* (1). Hence, both γ and π are smooth maps of constant rank. However, the composite map

$$\pi \circ \gamma\colon (0, 2\pi) \to \mathbb{R},\ t \mapsto \sin t$$

does not have constant rank, because its derivative

$$(\pi \circ \gamma)'\colon (0, 2\pi) \to \mathbb{R},\ t \mapsto -\cos t$$

vanishes for $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

Exercise 2:

(a) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Show that each of the projection maps $\pi_i\colon M_1 \times \ldots \times M_k \to M_i$ is a smooth submersion.

(b) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Choosing arbitrarily points $p_1 \in M_1, \ldots, p_k \in M_k$, for each $1 \leq j \leq k$ consider the map

$$\iota_j\colon M_j \to M_1 \times \ldots \times M_k,\ x \mapsto (p_1, \ldots, p_{j-1}, x, p_{j+1}, \ldots, p_k).$$

Show that each ι_j is a smooth embedding.

(c) Examine whether the following plane curves are smooth immersions:

(i) $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^3,\ t^2)$.

(ii) $\beta\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^3 - 4t,\ t^2 - 4)$.

If so, then examine also whether they are smooth embeddings.

(d) Show that the map

$$G\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u, v) \mapsto \big((2 + \cos 2\pi u)\cos 2\pi v,\ (2 + \cos 2\pi u)\sin 2\pi v,\ \sin 2\pi u\big)$$

is a smooth immersion.

Solution:

(a) Fix $i \in \{1, \dots, k\}$ and $p = (p_1, \dots, p_k) \in M_1 \times \dots \times M_k$. By *Exercise* 4*, Sheet* 3 we know that $\pi_i \colon M_1 \times \dots \times M_k \to M_i$ is a smooth map, while by *Exercise* 3*, Sheet* 4 we know that

$$\begin{aligned} T_p\big(M_1 \times \dots \times M_k\big) &\longrightarrow T_{p_1}M_1 \oplus \dots \oplus T_{p_i}M_i \oplus \dots \oplus T_{p_k}M_k \\ v &\mapsto \big(d(\pi_1)_p(v), \dots, d(\pi_i)_p(v), \dots, d(\pi_k)_p(v)\big) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism. Using the above identification, we infer that the differential of π_i at p,

$$d(\pi_i)_p \colon T_{p_1}M_1 \oplus \dots \oplus T_{p_i}M_i \oplus \dots \oplus T_{p_k}M_k \to T_{p_i}M_i,$$

is surjective. Since $p \in M_1 \times \dots \times M_k$ was arbitrary, we conclude that π_i is a smooth submersion.

(b) Fix $j \in \{1, \dots, k\}$ and points $p_1 \in M_1, \dots, p_{j-1} \in M_{j-1}, p_{j+1} \in M_{j+1}, \dots, p_k \in M_k$. We have already seen in the solution of *Exercise* 3*, Sheet* 4 that the map

$$\iota_j \colon M_j \to M_1 \times \dots \times M_k, \;\; x \mapsto (p_1, \dots, p_{j-1}, x, p_{j+1}, \dots, p_k)$$

is smooth, and it is also clear that ι_j is a homeomorphism onto its image

$$\iota_j(M_j) = \{p_1\} \times \dots \times \{p_{j-1}\} \times M_j \times \{p_{j+1}\} \times \dots \times \{p_k\}.$$

Moreover, given a point $p_j \in M_j$, using the identification

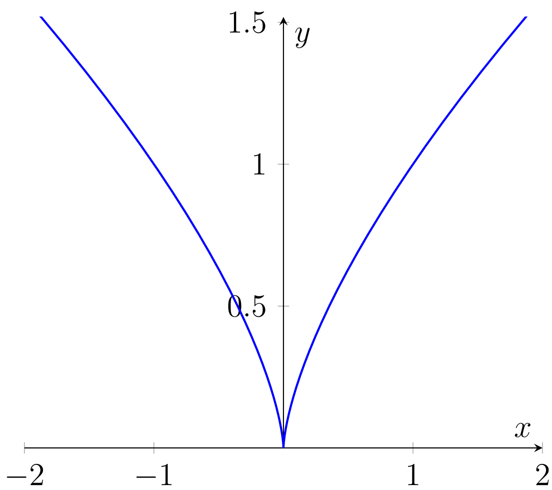
$$T_p\big(M_1 \times \dots \times M_k\big) \cong T_{p_1}M_1 \oplus \dots \oplus T_{p_i}M_i \oplus \dots \oplus T_{p_k}M_k,$$

where $p \coloneqq (p_1, \dots, p_{j-1}, p_j, p_{j+1}, \dots, p_k) \in M_1 \times \dots M_k$, we infer that the differential of ι_j at p,

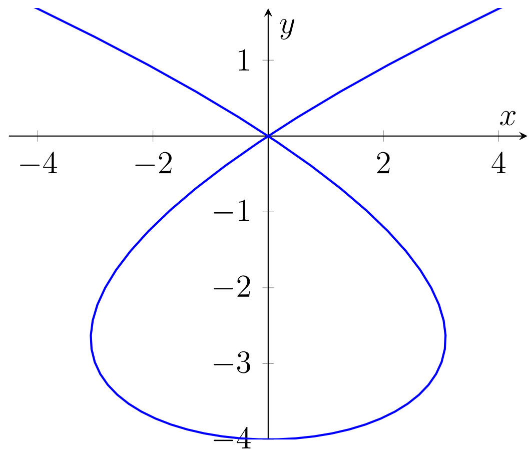
$$d(\iota_j)_{p_j} \colon T_{p_j}M_j \to T_{p_1}M_1 \oplus \dots \oplus T_{p_j}M_j \oplus \dots \oplus T_{p_k}M_k,$$

is injective. In conclusion, ι_j is a smooth embedding.

(c) We first deal with (i). The map $\alpha(t) = (t^3, t^2)$, $t \in \mathbb{R}$, is clearly smooth, but it is not an immersion, since $\alpha'(t) = (3t^2, 2t)$ vanishes at the point $t = 0$. Thus, α cannot be an embedding either.



We now deal with (ii). The map $\beta(t) = (t^3 - 4t, t^2 - 4)$, $t \in \mathbb{R}$, is clearly smooth and its velocity vector $\beta'(t) = (3t^2 - 4, 2t)$, $t \in \mathbb{R}$, is nowhere vanishing, so β is an injective immersion. However, the image curve $\beta(\mathbb{R})$ has a self-intersection for $t = -2$, $t = 2$, and hence β cannot be an embedding.



(d) The map G with component functions (G^1, G^2, G^3) is clearly smooth with Jacobian matrix

$$J_G(u,v) = \begin{pmatrix} \frac{\partial G^1}{\partial u}(u,v) & \frac{\partial G^1}{\partial v}(u,v) \\ \frac{\partial G^2}{\partial u}(u,v) & \frac{\partial G^2}{\partial v}(u,v) \\ \frac{\partial G^3}{\partial u}(u,v) & \frac{\partial G^3}{\partial v}(u,v) \end{pmatrix}$$

$$= \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi\big(2+\cos(2\pi u)\big)\sin(2\pi v) \\ -2\pi\sin(2\pi u)\sin(2\pi v) & 2\pi\big(2+\cos(2\pi u)\big)\cos(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}.$$

The 2×2 submatrix

$$\begin{pmatrix} \frac{\partial G^1}{\partial u}(u,v) & \frac{\partial G^1}{\partial v}(u,v) \\ \frac{\partial G^2}{\partial u}(u,v) & \frac{\partial G^2}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi\big(2+\cos(2\pi u)\big)\sin(2\pi v) \\ -2\pi\sin(2\pi u)\sin(2\pi v) & 2\pi\big(2+\cos(2\pi u)\big)\cos(2\pi v) \end{pmatrix}$$

of J_G has determinant

$$D_{12}(u,v) := -4\pi^2\big(2+\cos(2\pi u)\big)\sin(2\pi u),$$

the 2×2 submatrix

$$\begin{pmatrix} \frac{\partial G^1}{\partial u}(u,v) & \frac{\partial G^1}{\partial v}(u,v) \\ \frac{\partial G^3}{\partial u}(u,v) & \frac{\partial G^3}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi\big(2+\cos(2\pi u)\big)\sin(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}$$

of J_G has determinant

$$D_{13}(u,v) := 4\pi^2\big(2+\cos(2\pi u)\big)\cos(2\pi u)\sin(2\pi v),$$

and the 2×2 submatrix

$$\begin{pmatrix} \frac{\partial G^2}{\partial u}(u,v) & \frac{\partial G^2}{\partial v}(u,v) \\ \frac{\partial G^3}{\partial u}(u,v) & \frac{\partial G^3}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -2\pi\sin(2\pi u)\sin(2\pi v) & 2\pi\big(2+\cos(2\pi u)\big)\cos(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}$$

of J_G has determinant

$$D_{23}(u,v) := -4\pi^2\big(2+\cos(2\pi u)\big)\cos(2\pi u)\cos(2\pi v).$$

Observe now that for each $(u,v) \in \mathbb{R}^2$, at least one of the determinants $D_{12}(u,v)$, $D_{13}(u,v)$ and $D_{23}(u,v)$ is non-zero, since $\cos(2\pi\theta)$ and $\sin(2\pi\theta)$ do not vanish simultaneously. This implies that $\operatorname{rk}\big(J_G(u,v)\big) = 2$ for all $(u,v) \in \mathbb{R}^2$; see part (c) of *Exercise* 4, *Sheet* 2. In conclusion, G is a smooth immersion, as claimed.

Exercise 3 (Inverse function theorem for smooth manifolds)**:** Let $F\colon M \to N$ be a smooth map. Show that if $p \in M$ is a point such that the differential dF_p of F at p is invertible, then there exist connected neighborhoods U_0 of p in M and V_0 of $F(p)$ in N such that $F|_{U_0}\colon U_0 \to V_0$ is a diffeomorphism.

Before giving the solution to *Exercise* 3, we recall the following well known theorem:

Theorem (Inverse function theorem for open subsets of $\mathbb{R}^n$)**.** *Let $A \subseteq \mathbb{R}^n$ be open and consider a smooth function $F\colon A \to \mathbb{R}^n$. Suppose that there is a point $a \in A$ such that the Jacobian matrix of F at a is invertible. Then there exist connected open sets U and V such that $a \in U \subseteq A$ and $F(U) \subseteq V \subseteq \mathbb{R}^n$, for which the restriction $F|_U\colon U \to V$ admits a smooth inverse; that is, $F|_U$ is a diffeomorphism from U to V.*

Solution: The idea is to pass to a coordinate representation of F and to use the *inverse function theorem*. Let (U,φ) and (V,ψ) be charts for M and N around p and $F(p)$, respectively, such that $F(U) \subseteq V$, and assume WLOG that $\varphi(p) = 0$ and $\psi(F(p)) = 0$. Set $\widehat{U} := \varphi(U)$ and $\widehat{V} := \psi(V)$, and let

$$\widehat{F} = \psi \circ F \circ \varphi^{-1}\colon \widehat{U} \to \widehat{V}$$

be the coordinate representation of F, which is smooth with $\widehat{F}(0) = 0$. Since dF_p is invertible, the tangent space to M at p and to N at $F(p)$ must have the same dimension, and thus $\widehat{U}, \widehat{V} \subseteq \mathbb{R}^n$, where $n = \dim M = \dim N$. Observe now that the differential

$$d\widehat{F}_0 = d\psi_{F(p)} \circ dF_p \circ d(\varphi^{-1})_0$$

is invertible, because dF_p is invertible by assumption, and both $d(\varphi^{-1})_0$ and $d\psi_{F(p)}$ are invertible as well, as φ and ψ are diffeomorphisms. Note that the matrix representation of $d\widehat{F}_0$ with respect to the standard coordinates of $\mathbb{R}^n$ is the Jacobian of $\widehat{F}$ at 0. Therefore, by the *inverse function theorem* there are connected open neighborhoods $\widehat{U}_0 \subseteq \widehat{U}$ and $\widehat{V}_0 \subseteq \widehat{V}$ of 0 such that $\widehat{F}|_{\widehat{U}_0}\colon \widehat{U}_0 \to \widehat{V}_0$ is a diffeomorphism. Hence, for $U_0 := \varphi^{-1}(\widehat{U}_0) \ni p$ and $V_0 := \psi^{-1}(\widehat{V}_0) \ni F(p)$, the restriction $F|_{U_0}\colon U_0 \to V_0$ is a diffeomorphism, since we can write it as a composition of diffeomorphisms.

Remark. Exercise 3 has the following important corollary: a smooth map $F\colon M \to N$ is a local diffeomorphism if and only if dF_p is invertible for all $p \in M$. This also gives a very useful method to prove that some map is a diffeomorphism, without explicitly constructing a smooth inverse: a smooth bijective map $F\colon M \to N$ whose differential dF_p is invertible for all $p \in M$ is a diffeomorphism; see also *Exercise* 4(f).

Exercise 4: Prove the following assertions:

(a) Every composition of local diffeomorphisms is a local diffeomorphism.

(b) Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.

(c) Every local diffeomorphism is a local homeomorphism and an open map.

(d) The restriction of a local diffeomorphism to an open submanifold is a local diffeomorphism.

(e) Every diffeomorphism is a local diffeomorphism.

(f) Every bijective local diffeomorphism is a diffeomorphism.

(g) A map between smooth manifolds is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.

Solution: The key to the solution of this exercise is the above-mentioned corollary of the *inverse function theorem*: a smooth map is a local diffeomorphism if and only if its differential is everywhere invertible.

(a) The composition of two invertible linear maps is invertible.

(b) A block-diagonal matrix with invertible blocks on the diagonal is invertible. (Use *Exercise* 3, *Sheet* 4 for the implicit identification.)

(c) A diffeomorphism is clearly a homeomorphism, so a local diffeomorphism is a local homeomorphism. To see that a local homeomorphism $F\colon M \to N$ is open, let $U \subseteq M$ be an open set. For every $p \in U$, there exists $U_p \ni p$ such that $F(U_p)$ is open in N and $F|_{U_p}\colon U_p \to F(U_p)$ is a homeomorphism. In particular, the open subset $U \cap U_p \ni p$ of U is mapped to an open subset $F(U \cap U_p) \ni F(p)$ of $F(U_p)$. As $F(U_p)$ is itself open in N, we infer that $F(U \cap U_p)$ is open in N. Hence,

$$F(U) = \bigcup_{p \in U} F(U \cap U_p)$$

is open in N as well. In conclusion, F is an open map.

(d) The differential of the restriction is still invertible.

(e) We may take M and N as our open neighborhoods in the definition of a local diffeomorphism.

(f) The inverse map exists set theoretically, and it is smooth as it is smooth locally around each point by the *inverse function theorem*. Hence, the inverse map is smooth as well; in other words, the given map is a diffeomorphism.

(g) If some coordinate representation around each point is a (local) diffeomorphism, then the map is a local diffeomorphism by (a). The converse direction follows from the *inverse function theorem*.

Exercise 5: Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Prove the following assertions:

(a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(b) If $\dim M = \dim N$ and if F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Solution: Note that a local diffeomorphism is a smooth map by part (a) of *Exercise* 2*, Sheet* 3.

(a) Assume first that F is a local diffeomorphism. By part (d) of *Exercise* 1*, Sheet* 4 we infer that for any $p \in M$, the differential of F at p is an $\mathbb{R}$-linear isomorphism, and thus both injective and surjective. Hence, F is both a smooth immersion and a smooth submersion.

Assume now that F is both a smooth immersion and a smooth submersion. Then for every $p \in M$, its differential dF_p is both injective and surjective, and thus an $\mathbb{R}$-linear isomorphism. It follows from *Exercise* 3 that F is a local diffeomorphism.

(b) Since $\dim M = \dim N$, for any $p \in M$, the differential $dF_p\colon T_pM \to T_{F(p)}N$ is an $\mathbb{R}$-linear map between $\mathbb{R}$-vector spaces of the same dimension. Thus, dF_p is injective or surjective if and only if it an isomorphism. Therefore, F is a smooth immersion if and only if F is a smooth submersion, and hence (b) follows immediately from (a).

Exercise 6: Let M, N and P be smooth manifolds, and let $F\colon M \to N$ be a local diffeomorphism. Prove the following assertions:

(a) If $G\colon P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.

(b) If F is surjective and if $H\colon N \to P$ is any map, then H is smooth if and only if $H \circ F$ is smooth.

Solution: Recall that a local diffeomorphism is a smooth map by part (a) of *Exercise* 2*, Sheet* 3.

(a) If G is smooth, then $F \circ G$ is smooth by part (e) of *Exercise* 3*, Sheet* 3. Conversely, consider the smooth map $H := F \circ G\colon P \to N$ and fix a point $p \in P$. Since F is a local diffeomorphism, there exists an open neighborhood V of $G(p)$ such that $F(V)$ is open in N and $F|_V\colon V \to F(V)$ is a diffeomorphism. Since G is continuous by assumption, $U := G^{-1}(V)$ is an open subset of P, and since $G(p) \in V$, it holds that $p \in U$; in other words, U is an open neighborhood of p in P. Observe now that $G|_U = (F|_V)^{-1} \circ H|_U$ is smooth by part (e) of *Exercise* 3*, Sheet* 3, since $(F|_V)^{-1}$ is smooth by assumption and $H|_U$ is smooth by part (b) of *Exercise* 2*, Sheet* 3. It follows from part (a) of *Exercise* 2*, Sheet* 3 that G is smooth.

(b) If H is smooth, then $H \circ F$ is smooth by part (e) of *Exercise* 3*, Sheet* 3. Conversely, consider the smooth map $G := H \circ F$ and fix a point $q \in N$. Since F is surjective, there exists a point $p \in M$ such that $F(p) = q$, and since F is a local diffeomorphism, there exists an open neighborhood U of p such that $F(U)$ is open in N and $F|_U \colon U \to F(U)$ is a diffeomorphism; in particular, $F(U)$ is an open neighborhood of q in N. Observe now that $H|_{F(U)} = G|_U \circ (F|_U)^{-1}$ is smooth by part (e) of *Exercise* 3*, Sheet* 3, since $(F|_U)^{-1}$ is smooth by assumption and $G|_U$ is smooth by part (b) of *Exercise* 2*, Sheet* 3. It follows from part (a) of *Exercise* 2*, Sheet* 3 that H is smooth.