Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 8

Definition.

(a) Let X and Y be topological spaces. A (continuous) map $F\colon X \to Y$ is called *proper* if for every compact subset $K \subseteq Y$, the preimage $F^{-1}(K) \subseteq X$ is compact.

(b) Let M be a smooth manifold. An embedded submanifold S of M is said to be *properly embedded* if the inclusion $\iota\colon S \hookrightarrow M$ is a proper map.

Exercise 1:

(a) *Sufficient conditions for properness*: Let X and Y be topological spaces and let $F\colon X \to Y$ be a continuous map. Prove the following assertions:

  (i) If X is compact and Y is Hausdorff, then F is proper.

  (ii) If F is a topological embedding with closed image, then F is proper.

  (iii) If Y is Hausdorff and F has a continuous *left inverse*, i.e., a continuous map $G\colon Y \to X$ such that $G \circ F = \mathrm{Id}_X$, then F is proper.

(b) Let M be a smooth manifold and let S be an embedded submanifold of M. Show that S is properly embedded if and only if S is a closed subset of M.

(c) *Global graphs are properly embedded*: Let $f\colon M \to N$ be a smooth map between smooth manifolds. Show that the graph $\Gamma(f)$ of f is a properly embedded submanifold of $M \times N$.

Exercise 2:
Fix $n \geq 0$. Using

(i) the local slice criterion, and

(ii) the regular level set theorem,

show that $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$.

Exercise 3 (**to be submitted by Friday, 17.11.2023, 20:00**)**:**

(a) Consider the smooth function

$$f\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto x^3 + xy + y^3.$$

Show that if $c \in \mathbb{R} \setminus \{0, \frac{1}{27}\}$, then the level set $f^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.

(b) Consider the smooth function

$$\Phi\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto x^2 - y^2.$$

Given $c \in \mathbb{R}$, examine whether the level set $\Phi^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.

Exercise 4:
Let S be a subset of a smooth m-manifold M. Show that S is an embedded k-submanifold of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi\colon U \to \mathbb{R}^{m-k}$.

[Hint: Use the local slice criterion.]

Exercise 5:

(a) *Restricting the domain of a smooth map*: If $F\colon M \to N$ is a smooth map and if $S \subseteq M$ is an immersed or embedded submanifold, then the restriction $F|_S\colon S \to N$ is smooth.

(b) *Restricting the codomain of a smooth map*: Let M be a smooth manifold, let $S \subseteq M$ be an immersed submanifold, and let $G\colon N \to M$ be a smooth map whose image is contained in S. If G is a continuous map from N to S, then $G\colon N \to S$ is smooth.

(c) Let M be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Then every smooth map $G\colon N \to M$ whose image is contained in S is also smooth as a map from N to S.

Exercise 6:
Let M be a smooth manifold. Show that if S is an embedded submanifold of M, then there exists a unique topology and smooth structure on S such that the inclusion map $S \hookrightarrow M$ is a smooth embedding.

Exercise 7 (*Extension lemma for functions on submanifolds*)**:**
Let M be a smooth manifold, let $S \subseteq M$ be a smooth submanifold, and let $f \in C^\infty(S)$. Prove the following assertions:

(a) If S is an embedded submanifold, then there exists a neighborhood U of S in M and a smooth function $\widetilde{f}$ on U such that $\widetilde{f}|_S = f$.

[Hint: Use the local slice criterion and partitions of unity.]

(b) If S is a properly embedded submanifold, then the neighborhood U in (a) can be taken to be all of M.

[Hint: Take the construction in (a) and *Exercise* 1(b) into account.]