Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 9

Exercise 1:
Let M be a smooth manifold and let S be an immersed submanifold of M. Show that if any of the following conditions hold, then S is actually an embedded submanifold of M.

(a) The codimension of S in M is zero.

(b) The inclusion map $\iota\colon S \hookrightarrow M$ is proper.

(c) S is compact.

Exercise 2:
Let M be a smooth manifold. Show that if S is an immersed submanifold of M, then for the given topology on S, there exists a unique smooth structure on S such that the inclusion map $S \hookrightarrow M$ is a smooth immersion.

[Hint: Use part (b) of *Exercise* 5, *Sheet* 8.]

Exercise 3:

(a) Let M be a smooth manifold, let $S \subseteq M$ be an immersed or embedded submanifold, and let $p \in S$. Show that a vector $v \in T_pM$ is in T_pS if and only if there exists a smooth curve $\gamma\colon J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.

(b) Let M be a smooth manifold, let $S \subseteq M$ be an embedded submanifold and let $\gamma\colon J \to M$ be a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$.

Exercise 4:

(a) Let M be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Show that if $\Phi\colon U \to N$ is a local defining map for S, then it holds that

$$T_pS \cong \ker\left(d\Phi_p\colon T_pM \to T_{\Phi(p)}N\right) \quad \text{for every } p \in S \cap U.$$

(b) Let M be a smooth manifold. Suppose that $S \subseteq M$ is a level set of a smooth submersion $\Phi = (\Phi_1, \ldots, \Phi_k) \colon M \to \mathbb{R}^k$. Show that a vector $v \in T_pM$ is tangent to S if and only if $v\Phi_1 = \ldots = v\Phi_k = 0$.

Exercise 5:

(a) Consider the smooth curve

$$\beta \colon (-\pi, \pi) \to \mathbb{R}^2, \ t \mapsto (\sin 2t, \sin t)$$

from *Example 4.5*(2). Show that its image is not an embedded submanifold of $\mathbb{R}^2$.

[Be careful: this is not the same as showing that β is not a smooth embedding.]

(b) Consider the smooth function

$$\Phi \colon \mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto x^2 - y^2.$$

Show that the level set $\Phi^{-1}(0)$ is an immersed submanifold of $\mathbb{R}^2$.

[Hint: Set up an appropriate bijection and imitate the proof of *Proposition 5.13.*]

(c) Consider the smooth function

$$\Psi \colon \mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto x^2 - y^3.$$

Show that the level set $\Psi^{-1}(0)$ is not an immersed submanifold of $\mathbb{R}^2$.

[Hint: Argue by contradiction and use *Exercise* 3(a).]

Exercise 6 (to be submitted by Friday, 24.11.2023, 20:00):
Consider the smooth function

$$f \colon \mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto x^3 + y^3 + 1.$$

(a) Which are the regular values of f?

(b) For which $c \in \mathbb{R}$ is the level set $f^{-1}(c)$ an embedded submanifold of $\mathbb{R}^2$?

(c) Whenever the level set $S = f^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$, given $p \in S$, determine the tangent space $T_pS \cong d\iota_p(T_pS) \subset T_p\mathbb{R}^2 \cong \mathbb{R}^2$, where $\iota \colon S \hookrightarrow \mathbb{R}^2$ is the inclusion map.