CH. 5 : SUBMANIFOLDS

· __DEF. 5.1__ : Let M be a smooth mnfd. An __embedded submanifold of M__ is a subset $S \subseteq M$ that is a top. mnfd in the subspace topology, endowed with a smooth structure w.r.t. which the inclusion map $S \hookrightarrow M$ is a smooth embedding.

If S is an embedded submanifold of M, then the difference $\dim M - \dim S$ is called __the codimension of S in M__, and the containing mnfd M is called __the ambient manifold for S__.

(The empty set ϕ is an embedded submnfd of any dim.)

· __PROP. 5.2__ (Open submanifolds): Let M be a smooth mnfd. The embedded submnfds of codim 0 in M are exactly the open submnfds.

__PROOF__: If $U \subseteq M$ is an open submnfd (EX. 1.8(3)), then we have already seen that U is a smooth mnfd of $\dim U = \dim M$ and that the inclusion map $\iota : U \hookrightarrow M$ is a smooth embedding (EX. 4.4(3)), so $U \subseteq M$ is an embedded submnfd of codim 0.

Conversely, let $U \subseteq M$ be an embedded submnfd of codim 0. Then the inclusion $\iota : U \hookrightarrow M$ is a smooth embedding, and thus a local diffeomorphism by ESGE5(b), since $\dim U = \dim M$, so it is an open map by ESGE4(c). Therefore, U is an open subset of M. ∎

· **PROP. 5.3** (Images of embeddings as submanifolds): Let $F: N \to M$ be a smooth embedding and set $S := F(N)$. With the subspace top., S is a top. mnfd, and it has a unique smooth structure making it into an embedded submnfd of M with the property that F is a diffeomorphism onto its image.

PROOF: If we give S the subspace top. that it inherits from M, then the assumption that F is an embedding means that F can be considered as a homeo from N onto S, and thus S is a top. mnfd. We now give S a smooth structure by taking the smooth charts to be those of the form $(F(U), \varphi \circ F^{-1})$, where (U, φ) is a smooth chart for N; note that the smooth compatibility of these charts follows from the smooth compatibility of the corresponding charts for N. With this smooth structure on S, the map F is a diffeomorphism onto its image (essentially by dfn), and this is obviously the only smooth structure with this property. Finally, the inclusion map $\iota: S \hookrightarrow M$ is equal to the composition of a diffeomorphism followed by a smooth embedding

$$S \xrightarrow{F^{-1}} N \xrightarrow{F} M,$$

so it is a smooth embedding by ESG - E1(a)(iii). ■

Since every embedded submnfd is the image of a smooth embedding (namely its own inclusion map), PROP. 5.3 shows that embedded submnfds are exactly the images of smooth embeddings.

PROP. 5.4 (Graphs as submanifolds): Let M be a smooth m-mnfd, let N be a smooth n-mnfd, let $U \subseteq M$ be an open subset, and let $f: U \to N$ be a smooth map. Then the graph of f,

$$\Gamma(f) := \{ (x,y) \in M \times N \mid x \in U, y = f(x) \},$$

is an embedded m-dim submnfd of $M \times N$ diffeomorphic to U.

PROOF: (Recall EX. 1.3(1) and EX. 1.8(1)) Consider the map

$$\gamma_f : U \longrightarrow M \times N, \; x \longmapsto (x, f(x)).$$

It is a smooth map whose image is $\Gamma(f)$. Since the projection $\pi_M : M \times N \longrightarrow M$ satisfies $\pi_M \circ \gamma_f(x) = Id_U(x) = x$ for $x \in U$, the composition $d(\pi_M)_{(x,f(x))} \circ d(\gamma_f)_x$ is the identity on $T_x M$ for each $x \in U$. Thus, $d(\gamma_f)$ is injective, so γ_f is a smooth immersion. It is also a homeo onto its image, since $\pi_M|_{\Gamma(f)}$ is a continuous inverse for it. Thus, $\Gamma(f)$ is an embedded submnfd of $M \times N$ diffeomorphic to U by PROP. 5.3. ∎

In particular, if M and N are smooth mnfds, then for each $q \in N$ the subset $M \times \{q\}$, called a slice of the product mnfd, is an embedded submnfd of $M \times N$ diffeomorphic to M by PROP. 5.4 and ES3E3(b).

An embedded submnfd $S \subseteq M$ is said to be properly embedded if the inclusion $S \hookrightarrow M$ is a proper map. It will be shown in ES8E1(b) that an embedded submnfd $S \subseteq M$ is properly embedded iff S is a closed subset of M. Consequently, every compact embedded submnfd is properly embedded.

since compact subspaces of Hausdorff spaces are closed.

· **DEF. 5.5**:

(a) Given an open subset $U \subseteq \mathbb{R}^n$ and $k \in \{0, \dots, n\}$, a <u>k-dimensional slice of U</u> (or simply a k-slice) is any subset of the form

$$S = \{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \in U \mid x^{k+1} = c^{k+1}, \dots, x^n = c^n\}$$

for some constants $c^{k+1}, \dots, c^n \in \mathbb{R}$ (often taken to be zero). (When $k = 0$, then $S \subseteq \{pt\} \subseteq U$, while when $k = n$, then $S = U$. Sometimes it is convenient to consider slices defined by setting some subset of the coordinates other than the last ones equal to constants.) Note that every k-slice is homeomorphic to an open subset of $\mathbb{R}^k$.

(b) Let M be a smooth mnfd and let (U, φ) be a smooth chart for M. If S is a subset of U s.t. $\varphi(S)$ is a k-slice of $\varphi(U) \subseteq \mathbb{R}^n$, then we say that S is a <u>k-slice of U</u>.

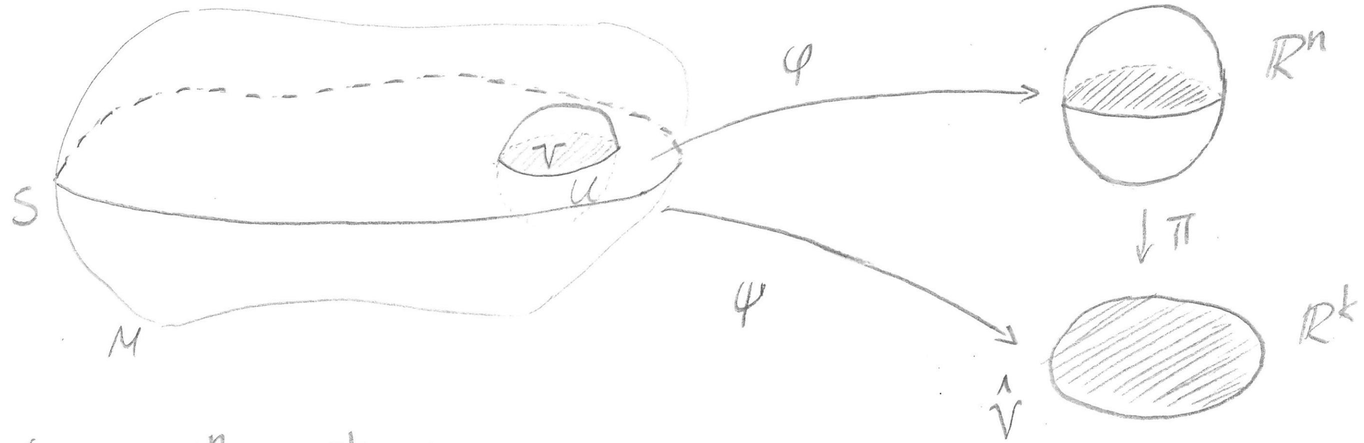
Given a subset $S \subseteq M$ and $k \in \mathbb{N}$, we say that S satisfies the <u>local k-slice condition</u> if each pt of S is contained in the domain of a smooth chart (U, φ) for M s.t. $S \cap U$ is a single k-slice in U. Any such chart is called a <u>slice chart for S in M</u>, and the corresponding coordinates $(x^1, \dots, x^n)$ are called <u>slice coordinates</u>.

· **THM 5.6** (Local slice criterion for embedded submanifolds): Let M be a smooth n-mnfd. If S is an embedded k-dimensional submnfd of M, then S satisfies the local k-slice condition. Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition, then with the subspace top., S is a top. mnfd of dim k, and it has a smooth structure making it into a k-dim embedded submnfd of M.

<u>PROOF:</u>

"$\Rightarrow$": Since the inclusion map $\iota : S \hookrightarrow M$ is in particular a smooth immersion, by the rank theorem we infer that for any $p \in S$ there are smooth charts (U, φ) for S and (V, ψ) for M, both centered at p, in which the inclusion map $\iota|_U : U \hookrightarrow V$ has the coordinate representation $(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$. Now, choose $0 < \varepsilon \ll 1$ so that both U and V contain coordinate balls $U_0 \subseteq U$ and $V_0 \subseteq V$ of radius $\varepsilon > 0$ centered at p. It follows that $U_0 \cong \iota(U_0)$ is exactly a single slice in V_0 (using the above local descriptions). Since $S \subseteq M$ has the subspace top. and since U_0 is open in S, there is an open subset $W \subseteq M$ s.t. $U_0 = W \cap S$. Setting $V_1 := W \cap V_0$, we obtain a smooth chart $(V_1, \psi|_{V_1})$ for M containing p s.t. $V_1 \cap S = U_0 \cap V_0 = U_0$, which is a single slice of V_1 (as U_0 is a single slice of V_0).

"$\Leftarrow$": With the subspace top., S is Hausdorff and second-countable, because both properties are inherited by subspaces. T.s.t. S is locally Euclidean, we construct an atlas. (The idea of the construction is that if $(x^1, \dots, x^n)$ are slice coordinates for S in M, then we can use $(x^1, \dots, x^k)$ as local coordinates for S.)



Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first k-coordinates. Let (U, φ) be a slice chart for S in M, and define $V := U \cap S$, $\hat{V} := (\pi \circ \varphi)(V)$, $\psi := \pi \circ \varphi|_V : V \to \hat{V}$. By dfn of slice charts, $\varphi(V)$ is the intersection of $\varphi(U)$ with a certain k-slice $A \subseteq \mathbb{R}^n$ defined by setting $x^{k+1} = c^{k+1}, \dots, x^n = c^n$, and thus $\varphi(V)$ is open in A. Since $\pi|_A : A \to \mathbb{R}^k$ is a diffeomorphism, it follows that $\hat{V}$ is open in $\mathbb{R}^k$. Moreover, ψ is a homeo, because it has a cont. inverse given by $\varphi^{-1} \circ j|_{\hat{V}}$, where $j : \mathbb{R}^k \to \mathbb{R}^n$ is

$$j(x^1, \dots, x^k) = (x^1, \dots, x^k, c^{k+1}, \dots, c^n).$$

Thus, S is a top. k-mnfd, and the inclusion map $\iota : S \hookrightarrow M$ is clearly a top embedding.

We now check that the charts constructed above are smoothly compatible. Let (U, φ) and (U', φ') be two slice charts for S in M and let (V, ψ) and (V', ψ') be the corresponding charts for S. The transition map is given by $\psi' \circ \psi^{-1} = \pi \circ \varphi' \circ \varphi^{-1} \circ j$, which is smooth as a composite of four smooth maps. Hence, the atlas we have constructed is actually a smooth atlas (see REM on p. 7), and it defines a smooth structure on S. In terms of a slice chart (U, φ) for M and the corresponding chart (V, ψ) for S, $\iota : S \hookrightarrow M$ has a coord repr. of the form $(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, c^{k+1}, \dots, x^n)$, so it is a smooth immersion, and we are done by the previous paragraph.

Notice that the local slice condition for $S \subseteq M$ is a condition on the subset S only; it does not presuppose any particular topology or smooth structure on S. According to ES8E6, the smooth mnfd structure constructed in THM 5.6 is the unique one in which S can be considered as a submnfd, so a subset satisfying the local slice condition is an embedded submnfd in only one way.

· DEF. 5.7: Let $\Phi: M \to N$ be a map. If $c \in N$, then $\Phi^{-1}(c)$ is called a level set of Φ. (In the special case $N = \mathbb{R}^k$ and $c = 0$, the level set $\Phi^{-1}(0)$ is usually called the zero set of Φ.)

Assume now that Φ is a smooth map. A pt $p \in M$ is called a regular pt of Φ if $d\Phi_p: T_pM \to T_{\Phi(p)}N$ is surjective; otherwise, we say that p is a critical pt of Φ. A pt $c \in N$ is called a regular value of Φ if every pt of the level set $\Phi^{-1}(c)$ is a regular pt; otherwise, we say that c is a critical value of Φ. (In particular, if $\Phi^{-1}(c) = \emptyset$, then c is a regular value.) Finally, a level set $\Phi^{-1}(c)$ is called a regular level set if c is a regular value of Φ.

· REM. 5.8: Let $\Phi: M \to N$ be a smooth map.

1) If $\dim M < \dim N$, then every pt of M is a critical pt of Φ.

2) Every pt of M is regular iff Φ is a smooth submersion.

3) By LEM. 4.3, the set of regular pts of Φ is an open subset of M (but may be empty).

Consider the three smooth fncts

$$\Theta : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (x,y) \longmapsto x^2 - y,$$
$$\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (x,y) \longmapsto x^2 - y^2,$$
$$\Psi : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (x,y) \longmapsto x^2 - y^3.$$

Although the zero set $\Theta^{-1}(0)$ of Θ is an embedded submnfd of $\mathbb{R}^2$, it will be shown in ES8E3(b) and ES9E4(c) that neither the zero set $\Phi^{-1}(0)$ of Φ nor the zero set $\Psi^{-1}(0)$ of Ψ is an embedded submnfd of $\mathbb{R}^2$. Hence, it is fairly easy to find level sets of smooth fncts that are not smooth submnfd. In fact, without further assumptions on the smooth fnct, the situation is about as bad as could be imagined: according to THM 2.16, every closed subset of M can be expressed as the zero set of a smooth non-negative real-valued fnct.

· THM 5.9 (Constant-rank level set theorem): Let $\Phi : M \longrightarrow N$ be a smooth map of constant rank r. Each level set of Φ is a properly embedded submnfd of codim r in M.

In particular, if Φ is a smooth submersion, then each level set of Φ is a properly embedded submnfd of M of codim $r = \dim N$.

PROOF: Set $m := \dim M$, $n := \dim N$ and $k := m - r$. Pick $c \in N$ and set $S := \Phi^{-1}(c)$. By the rank theorem, for each $p \in S$ there are smooth charts (U, φ) centered at p and (V, ψ) centered at $c = \Phi(p)$ in which Φ has a coordinate representation of the form

$$\Phi(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0),$$

and hence $S \cap U = \Phi^{-1}(0) \cap U$ is the slice

$$\{(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) \in U \mid x^1 = \ldots = x^r = 0\}.$$

Therefore, S satisfies the local $(k = m-r)$-slice condition, so it is an embedded submnfd of dim k by THM 5.6. It is closed in M by continuity of Φ, so it is properly embedded by ES8E1(b). ∎

· <u>COR. 5.10</u> (Regular level set theorem): Every regular level set of a smooth map between smooth mnfds is a properly embedded submnfd whose codim is equal to the dim of the codomain.

<u>PROOF</u>: Let $\Phi: M \to N$ be a smooth map and let $c \in N$ be a regular value of Φ. By LEM 5.3, the set

$$U = \{p \in M \mid \operatorname{rk}(d\Phi_p) = \dim N\} \subseteq M$$

is open in M, and contains $\Phi^{-1}(c)$ by assumption. Thus, $\Phi|_U : U \to N$ is a smooth submersion, so $\Phi^{-1}(c)$ is an embedded submnfd of U by THM 5.9. It follows now from PROP. 5.2 and ES6E1(a)(iii) that

$$\Phi^{-1}(c) \hookrightarrow U \hookrightarrow M$$

is a smooth embedding, so $\Phi^{-1}(c)$ is an embedded submnfd of M, and it is closed (so properly embedded by ES8E1(b)) by continuity. ∎

Not all embedded submnfds can be expressed as level sets of smooth submersions. However, the next proposition shows that every embedded submnfd is at least locally of this form.

· PROP. 5.11 : Let S be a subset of a smooth m-mnfd M. Then S is an embedded k-submnfd of M iff every pt of S has a neighborhood U in M s.t. $U \cap S$ is a level set of a smooth submersion.

PROOF : ES8E4