Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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# Exercise Sheet 9 – Solutions

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**Exercise 1:** Let $M$ be a smooth manifold and let $S$ be an immersed submanifold of $M$. Show that if any of the following conditions hold, then $S$ is actually an embedded submanifold of $M$.

(a) The codimension of $S$ in $M$ is zero.

(b) The inclusion map $\iota\colon S \hookrightarrow M$ is proper.

(c) $S$ is compact.

**Solution:** Since $S$ is an immersed submanifold of $M$, the inclusion map $\iota\colon S \hookrightarrow M$ is an injective smooth immersion. If any of the above conditions holds, then *Proposition 4.6* implies that $\iota$ is a smooth embedding; in particular, $\iota(S) = S$ is endowed with the subspace topology inherited from $M$. Therefore, in any of these three cases, $S$ is an embedded submanifold of $M$.

**Exercise 2:** Let $M$ be a smooth manifold. Show that if $S$ is an immersed submanifold of $M$, then for the given topology on $S$, there exists a unique smooth structure on $S$ such that the inclusion map $S \hookrightarrow M$ is a smooth immersion.
[Hint: Use part (b) of *Exercise* 5, *Sheet* 8.]

**Solution:** Denote by $\iota$ the inclusion map $S \hookrightarrow M$ of the immersed submanifold $S$ of $M$ and by $\widetilde{S}$ the topological space $S$ endowed now with another smooth structure such that the inclusion map $\widetilde{\iota}\colon \widetilde{S} \hookrightarrow M$ is a smooth immersion. Note that $\widetilde{S}$ is an immersed submanifold of $M$. Since $S$ and $\widetilde{S}$ have the same topology by assumption, both maps $\iota\colon S \to \widetilde{S}$ and $\widetilde{\iota}\colon \widetilde{S} \to S$ are continuous, and hence smooth by part (b) of *Exercise* 5, *Sheet* 8. Therefore, $S$ is diffeomorphic to $\widetilde{S}$.

*Remark.* It is possible for a given subset $S$ of a smooth manifold $M$ to have more than one topology making it into an immersed submanifold of $M$. However, for *weakly embedded submanifolds*[1], we have the following uniqueness result, which can be proved similarly to

[1]We refer to the *Remark* after the solution of *Exercise* 5, *Sheet* 8 for the definition of this notion

*Exercise* 2: *If $M$ is a smooth manifold and if $S$ is a weakly embedded submanifold of $M$, then $S$ has only one topology and smooth structure with respect to which it is an immersed submanifold of $M$.*

**Exercise 3:**

(a) Let $M$ be a smooth manifold, let $S \subseteq M$ be an immersed or embedded submanifold, and let $p \in S$. Show that a vector $v \in T_pM$ is in $T_pS$ if and only if there exists a smooth curve $\gamma\colon J \to M$ whose image is contained in $S$, and which is also smooth as a map into $S$, such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.

(b) Let $M$ be a smooth manifold, let $S \subseteq M$ be an embedded submanifold and let $\gamma\colon J \to M$ be a smooth curve whose image happens to lie in $S$. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$.

**Solution:**

(a) Assume that the given vector $v \in T_pM$ lies also in $T_pS$, which is identified with $d\iota_p(T_pS)$, so that $v = d\iota_p(w)$ for some $w \in T_pS$. By part (a) of *Exercise* 5*, Sheet* 4 there exists a smooth curve $\gamma\colon J \to S$ such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = w$. Since $S$ is an immersed (or embedded) submanifold of $M$, the inclusion map $\iota\colon S \hookrightarrow M$ is a smooth immersion, so the composite map $\iota \circ \gamma\colon J \to M$ is a smooth curve in $M$ whose image is clearly contained in $S$, which satisfies $0 \in J$, $(\iota \circ \gamma)(0) = p$, and finally by part (b) of *Exercise* 5*, Sheet* 4 we also have

$$(\iota \circ \gamma)'(0) = d\iota_{\gamma(0)}\big(\gamma'(0)\big) = d\iota_p(w) = v.$$

The converse follows immediately from part (a) of *Exercise* 5*, Sheet* 4 in view of the identification of $T_pS$ with $d\iota_p(T_pS)$.

(b) By assumption and by part (c) of *Exercise* 5*, Sheet* 8 the given map $\gamma$ is also smooth as a map from $J$ to $S$, so the statement follows immediately from part (a).

**Exercise 4:**

(a) Let $M$ be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Show that if $\Phi\colon U \to N$ is a local defining map for $S$, then it holds that

$$T_pS \cong \ker\big(d\Phi_p\colon T_pM \to T_{\Phi(p)}N\big) \quad \text{for every } p \in S \cap U.$$

(b) Let $M$ be a smooth manifold. Suppose that $S \subseteq M$ is a level set of a smooth submersion $\Phi = (\Phi_1, \ldots, \Phi_k)\colon M \to \mathbb{R}^k$. Show that a vector $v \in T_pM$ is tangent to $S$ if and only if $v\Phi_1 = \ldots = v\Phi_k = 0$.

**Solution:**

(a) Recall that we identify $T_pS$ with its image $d\iota_p(T_pS) \subseteq T_pM$, where $\iota\colon S \hookrightarrow M$ is the inclusion map, which is a smooth embedding by assumption. Note that by hypothesis we have $S \cap U = \Phi^{-1}(q)$ for some $q \in N$. Therefore, we have $\Phi \circ \iota|_{S\cap U} = c_q$, where

$c_q\colon S\cap U \to N$ is the constant map on $S\cap U$ with value $q\in N$. Thus, if $p\in S\cap U$ is arbitrary, then

$$0 = d(c_q)_p = d\Phi_p \circ d(\iota|_{S\cap U})_p.$$

Hence the differential $d(\iota|_{S\cap U})_p$ induces an injective map from $T_pS$ to $\ker d\Phi_p$ (because $\iota$ is an embedding).

In order to conclude, it suffices to show that both spaces have the same dimension. Denote by $m$, $n$, $s$ the dimension of $M$, $N$, $S$, respectively. By *Corollary 5.10* the codimension of $S$ in $M$ is $n$, i.e., $m-s=n$. On the other hand, by linear algebra and by the surjectivity of $d\Phi_p$ we have

$$n = \dim \operatorname{im} d\Phi_p = \underbrace{\dim T_pM}_{=m} - \dim\ker d\Phi_p \implies$$
$$\implies \dim\ker d\Phi_p = m-n = s.$$

Hence, $T_pS$ and $\ker d\Phi_p$ have the same dimension $s$, and are thus identified via $d\iota_p$.

(b) Fix $p\in S$. By part (a) we know that $v\in T_pM$ is tangent to $S$ if and only if $d\Phi_p(v)=0$. Denote by $\mathrm{pr}_1,\dots,\mathrm{pr}_k\colon \mathbb{R}^k\to\mathbb{R}$ the projection maps to the corresponding coordinates. By the description of $T_p\mathbb{R}^k$, note that a vector $w\in T_p\mathbb{R}^k$ is 0 if and only if $w(\mathrm{pr}_i)=0$ for all $1\le i\le k$. Hence,

$$d\Phi_p(v)=0 \iff d\Phi_p(v)(\mathrm{pr}_i)=0,\ \forall 1\le i\le k \iff v(\underbrace{\mathrm{pr}_i\circ\Phi}_{=\Phi_i})=0,\ \forall 1\le i\le k,$$

which completes the proof of (b).

**Exercise 5:**

(a) Consider the smooth curve

$$\beta\colon(-\pi,\pi)\to\mathbb{R}^2,\ t\mapsto(\sin 2t,\,\sin t)$$

from *Example 4.5*(2). Show that its image is not an embedded submanifold of $\mathbb{R}^2$.

(b) Consider the smooth function

$$\Phi\colon\mathbb{R}^2\to\mathbb{R},\ (x,y)\mapsto x^2-y^2.$$

Show that the level set $\Phi^{-1}(0)$ is an immersed submanifold of $\mathbb{R}^2$.

[Hint: Set up an appropriate bijection and imitate the proof of *Proposition 5.13*.]

(c) Consider the smooth function

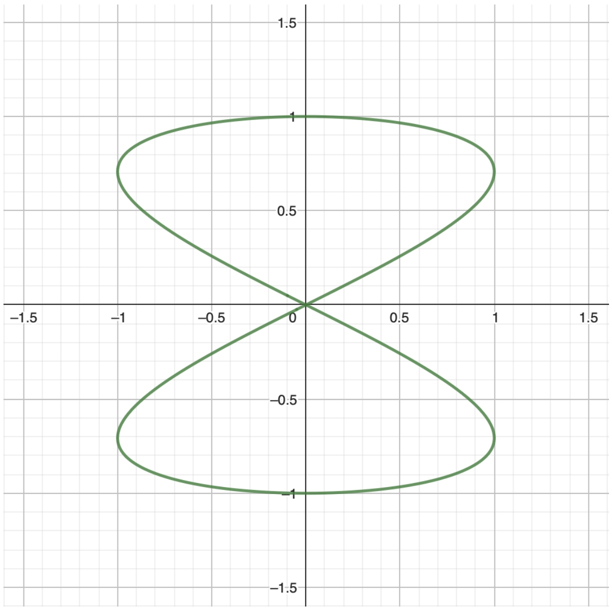
$$\Psi\colon\mathbb{R}^2\to\mathbb{R},\ (x,y)\mapsto x^2-y^3.$$

Show that the level set $\Psi^{-1}(0)$ is not an immersed submanifold of $\mathbb{R}^2$.

[Hint: Argue by contradiction and use *Exercise* 3(a).]

**Solution:**

(a) The image of $\beta$ has been plotted below.

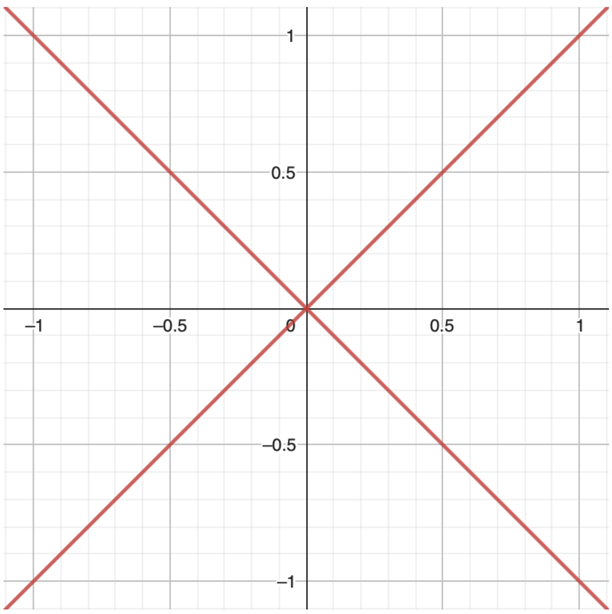



Endowed with the subspace topology inherited from $\mathbb{R}^2$, the image of $\beta$ is not a topological manifold. Indeed, essentially the same argument as the one presented in the solution of *Exercise* 4, *Sheet* 1 shows that $\beta(-\pi, \pi)$ is not locally Euclidean at the (self-intersection) point $(0, 0) \in \beta(-\pi, \pi)$. Therefore, the image of $\beta$ cannot be an embedded submanifold of $\mathbb{R}^2$.

(b) The level set

$$\begin{aligned} \Phi^{-1}(0) &= \big\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0\big\} \\ &= \big\{(x, y) \in \mathbb{R}^2 \mid (y - x)(y + x) = 0\big\} \end{aligned}$$

has been plotted below.



Even though it is not an embedded submanifold of $\mathbb{R}^2$, as already demonstrated in the solution to part (b) of *Exercise* 3, *Sheet* 8, we will show that $\Phi^{-1}(0)$ can be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$. To this end, note that there is a bijection between $\Phi^{-1}(0)$ and the subset $S := S_0 \sqcup S_1$ of $\mathbb{R}^2$, where

$$S_0 := \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} \cong \mathbb{R}$$

and

$$S_1 := \{(x, 1) \in \mathbb{R}^2 \mid x \in \mathbb{R} \setminus \{0\}\} \cong \mathbb{R} \setminus \{0\}.$$

As $S_0$ is the graph of the constant function $\mathbb{R} \to \mathbb{R}$, $x \mapsto 0$, and $S_1$ is the graph of the constant function $\mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto 1$, they are both embedded submanifolds of $\mathbb{R}^2$, and thus so is $S = S_1 \sqcup S_2$; in particular, the inclusion map $\iota\colon S \hookrightarrow \mathbb{R}^2$ is a smooth embedding. Using the bijection $G\colon S \to \Phi^{-1}(0)$, we endow $\Phi^{-1}(0)$ with a topology by declaring a subset $X \subseteq \Phi^{-1}(0)$ to be open if and only if $G^{-1}(X) \subseteq S$ is open, and with a smooth structure by taking the smooth charts for $\Phi^{-1}(0)$ to be those of the form $\big(G(U), \varphi \circ G^{-1}\big)$, where $(U, \varphi)$ is a smooth chart for $S$. With this topology (which is different from the subspace topology) and smooth structure, $S$ is a smooth manifold and $G$ is a diffeomorphism. Since the inclusion map $\Phi^{-1}(0) \hookrightarrow \mathbb{R}^2$ can be written as the composition

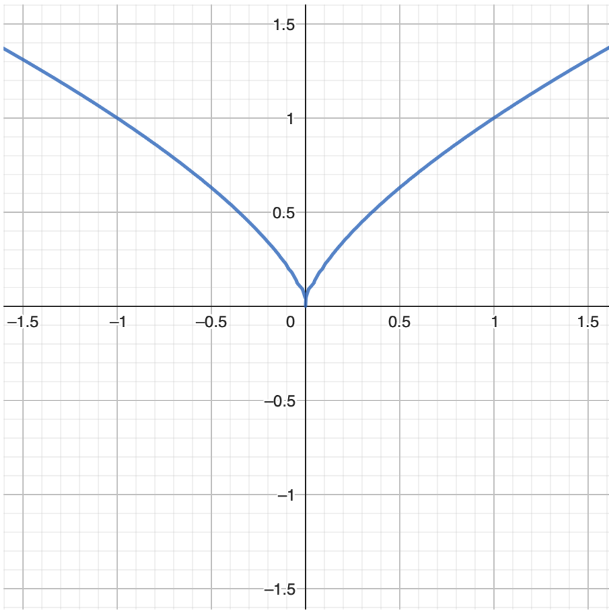
$$\Phi^{-1}(0) \xrightarrow{G^{-1}} S \xrightarrow{\iota} \mathbb{R}^2$$

of a diffeomorphism followed by a smooth immersion, it is itself is a smooth immersion by *Exercise 1*(a)(ii) and *Exercise 5*(a) from *Exercise Sheet 6*. In conclusion, $\Phi^{-1}(0)$ is an immersed submanifold of $\mathbb{R}^2$.

(c) The level set

$$\Psi^{-1}(0) = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^3 = 0\}$$

has been plotted below.




We assume that $\Psi^{-1}(0)$ can be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$ and we will derive a contradiction using *Exercise* 3(a). To this end, observe that $\Psi^{-1}(0)$ must be 1-dimensional; indeed, $\Psi^{-1}(0)\setminus\{(0,0)\}$ is an embedded 1-submanifold of $\mathbb{R}^2$, as its two connected components, corresponding to $(x,y)\in\Phi^{-1}(0)$ with $x<0$ (the left branch) and $(x,y)\in\Phi^{-1}(0)$ with $x>0$ (the right branch), are the graphs of the smooth functions $x\in(-\infty,0)\mapsto x^{\frac{2}{3}}$ and $x\in(0,+\infty)\mapsto x^{\frac{2}{3}}$, respectively. Therefore, $T_{(0,0)}\Phi^{-1}(0)$ is a 1-dimensional subspace of $T_{(0,0)}\mathbb{R}^2\cong\mathbb{R}^2$, so by *Exercise* 3(a) there exists a smooth curve $\gamma\colon(-\varepsilon,\varepsilon)\to\mathbb{R}^2$ whose image lies in $\Phi^{-1}(0)$ and which satisfies $\gamma(0)=(0,0)$ and $\gamma'(0)\neq 0$. Writing $\gamma(t)=\big(x(t),y(t)\big)$, we see that $y(t)$ takes a global minimum at $t=0$, so $y'(0)=0$. On the other hand, since $\gamma(t)\in\Phi^{-1}(0)$ for every $t\in(-\varepsilon,\varepsilon)$, we have $x^2(t)=y^3(t)$ for every $t\in(-\varepsilon,\varepsilon)$. Differentiating twice and setting $t=0$, we obtain $x'(0)=0$, and since $y'(0)=0$, we conclude that $\gamma'(0)=0$, which is a contradiction. Hence, the level set $\Psi^{-1}(0)$ is not an immersed submanifold of $\mathbb{R}^2$.

**Exercise 6:** Consider the smooth function

$$f\colon\mathbb{R}^2\to\mathbb{R},\ (x,y)\mapsto x^3+y^3+1.$$

(a) Which are the regular values of $f$?

(b) For which $c\in\mathbb{R}$ is the level set $f^{-1}(c)$ an embedded submanifold of $\mathbb{R}^2$?

(c) Whenever the level set $S=f^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$, given $p\in S$, determine the tangent space $T_pS\cong d\iota_p(T_pS)\subset T_p\mathbb{R}^2\cong\mathbb{R}^2$, where $\iota\colon S\hookrightarrow\mathbb{R}^2$ is the inclusion map.

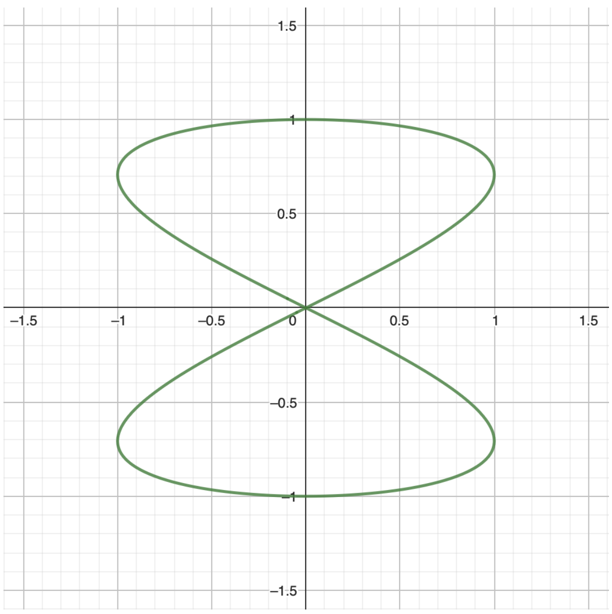
**Solution:**

(a) The gradient of $f$ at an arbitrary point $(x,y)\in\mathbb{R}^2$ is given by

$$\operatorname{grad}(f)(x,y)=(3x^2,\,3y^2),$$

and it is obvious that it vanishes precisely at the origin $(x,y)=(0,0)\in\mathbb{R}^2$. Since $f(0,0)=1$ and since the fibers of $f$ are disjoint, we conclude that every $c\in\mathbb{R}\setminus\{1\}$ is a regular value of $f$, while $c=1$ is a critical value of $f$.

The level sets $f^{-1}(-9)$ (in green), $f^{-1}(1)$ (in purple) and $f^{-1}(9)$ (in red) have been plotted below:



(b) By *Corollary 5.10* we infer that each level set $f^{-1}(c)$, where $c \neq 1$, is a properly embedded submanifold of $\mathbb{R}^2$. Now, regarding the level set $f^{-1}(1)$, *Corollary 5.10* does *not* say that $f^{-1}(1)$ is not an embedded submanifold, so we have to argue differently in order to treat this case. Observe that

$$f^{-1}(1) = \big\{(x, y) \in \mathbb{R}^2 \mid x^3 + y^3 = 0\big\} = \big\{(x, -x) \mid x \in \mathbb{R}\big\}$$

is the line $y = -x$ in the plane $\mathbb{R}^2$ (plotted in purple above), which is clearly diffeomorphic to $\mathbb{R}$, and hence (it is straightforward to check that) $f^{-1}(1)$ is a properly embedded submanifold of $\mathbb{R}^2$, taking also part (b) of *Exercise* 1*, Sheet* 8 into account.

In conclusion, all level sets of $f$ are properly embedded submanifolds of $\mathbb{R}^2$.

(c) Let $c \in \mathbb{R} \setminus \{1\}$, set $S := f^{-1}(c)$, and pick $p = (p_x, p_y) \in S$. By part (b) and by *Exercise* 4(a) we know that $T_pS = \ker df_p$, and the differential $df_p$ is represented by the row matrix $(3p_x^2, 3p_y^2)$. Thus, if $V = (V_x, V_y) \in T_p\mathbb{R}^2 \cong \mathbb{R}^2$, then

$$df_p(V_x, V_y) = 3\, p_x^2\, V_x + 3\, p_y^2\, V_y,$$

and hence

$$T_pS = \big\{V = (V_x, V_y) \in T_p\mathbb{R}^2 \mid p_x^2\, V_x + p_y^2\, V_y = 0\big\}.$$

Finally, recall that

$$S := f^{-1}(1) = \big\{(x, y) \in \mathbb{R}^2 \mid x + y = 0\big\},$$

which is a linear subspace of $\mathbb{R}^2$ (e.g., $S$ may be viewed as the kernel of the linear map $L\colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y)^T \mapsto (1, 1) \cdot (x, y)^T = x + y$), and hence

$$T_pS = \big\{V = (V_x, V_y) \in T_p\mathbb{R}^2 \mid V_x + V_y = 0\big\}$$

for any $p \in S$ (e.g., by applying *Exercise* 4(a) to the (smooth) linear map $L$ described above).