CH. 6 : VECTOR BUNDLES

In Ch. 3 we saw that the tangent bundle of a smooth mnfd has a natural structure as a smooth mnfd in its own right. The natural coordinates we constructed on TM make it look locally like the Cartanian product of an open subset of M^n with $\mathbb{R}^n$. This kind of structure arises quite frequently - a collection of vector spaces, one for each pt in M, glued together in a way that looks locally like the Cartesian product of M with $\mathbb{R}^n$, but globally may be "twisted". Such structures are called vector bundles, and will be discussed briefly here.

· <u>DEF. 6.1</u>: Let M be a top. sp. A <u>real vector bundle of rank k over M</u> is a top. sp. E together with a cont. surj. map $\pi: E \longrightarrow M$ satisfying the following conditions:

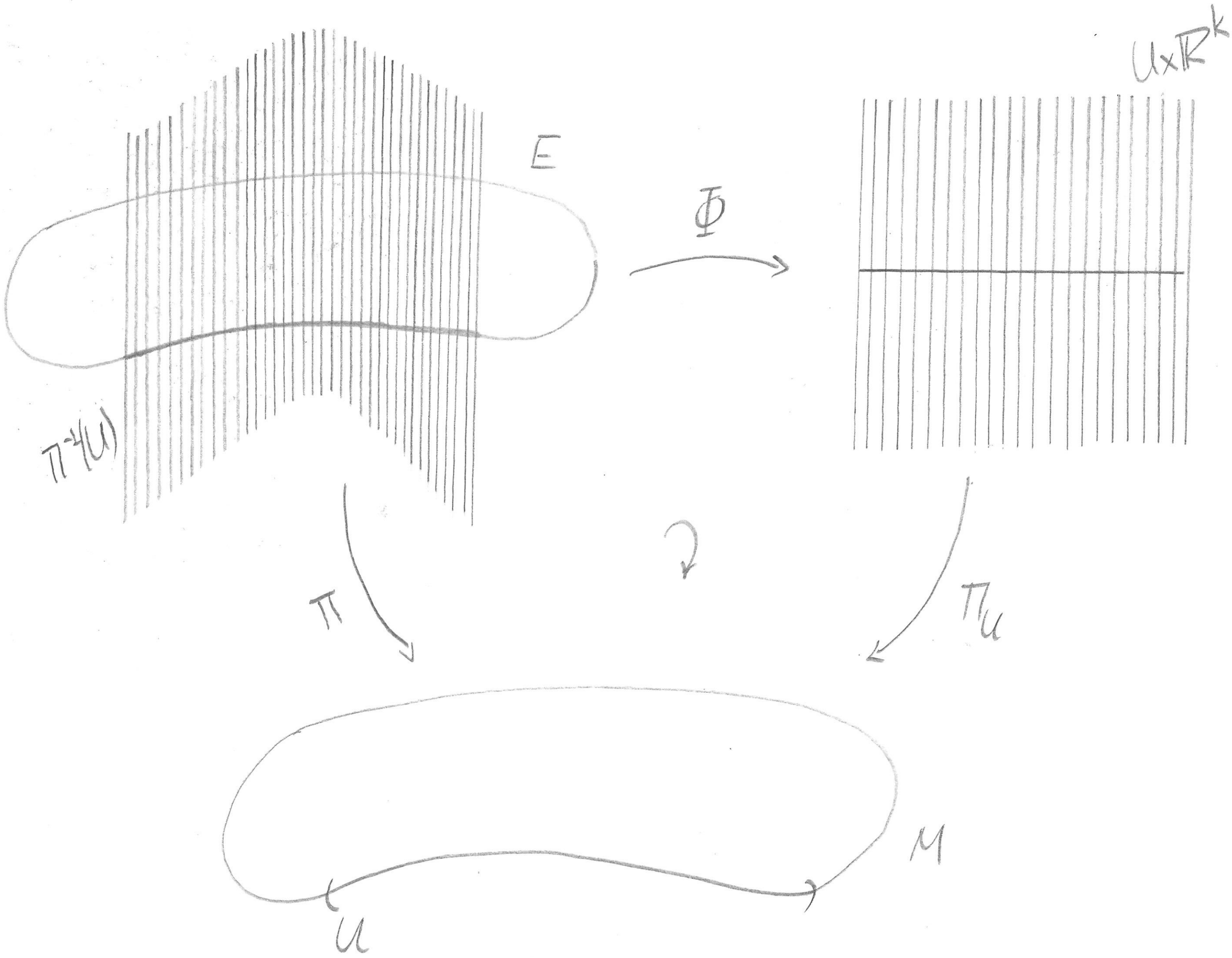
(i) For each $p \in M$, the <u>fiber</u> $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dim $\mathbb{R}$-v.s.

(ii) For each $p \in M$, there exists an open neighborhood U of p in M and a homeomorphism $\Phi: \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$, called a <u>local trivialization of E over U</u>, satisfying the following conditions:

- $\pi_U \circ \Phi = \pi$, where $\pi_U : U \times \mathbb{R}^k \longrightarrow U$ is the projection
- for each $q \in U$, the restriction of Φ to E_q is an $\mathbb{R}$-v.s. isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

The space E is called <u>the total space</u> of the bundle,

M is called its base, and π is its projection.



· DEF. 6.2: With the same notation as in DEF. 6.1, if both M and E are smooth mnfds, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, then E is called a smooth vector bundle over M. In this case, any local trivialization that is a diffeomorphism onto its image is called a smooth local trivialization.

· DEF. 6.3: With the same notation as in DEF. 6.1, if there exists a local trivialization of E over all of M, called a global trivialization of E, then E is called a trivial bundle.

If $E \to M$ is a smooth vector bundle that admits a smooth global trivialization, then we say that E is <u>smoothly trivial</u>. In this case, E is diffeomorphic to $M \times \mathbb{R}^k$, not just homeomorphic (as in the previous case).

· <u>EXAMPLE 6.4</u>: Given any top. sp. M, the product space $E := M \times \mathbb{R}^k$ with $\pi = \pi_M : M \times \mathbb{R}^k \to M$ as its projection is a rk k vector bundle over M. Any such bundle, called a <u>product bundle</u>, is trivial (with the identity map $\Phi = Id_E : M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ as a global trivialization). If M is a smooth mnfd, then the (smooth) product bundle $M \times \mathbb{R}^k$ is smoothly trivial.

· <u>PROP. 6.5</u> (The tangent bundle as a vector bundle): Let M be a smooth n-mnfd and let TM be its tangent bundle. With its standard projection map $\pi : TM \to M$, its natural vector space structure on each fiber, and the topology and smooth structure constructed in PROP. 3.19, $\pi : TM \to M$ is a smooth vector bundle of rank n over M.

<u>PROOF</u>: Given any smooth chart (U, φ) for M with coordinate fncts (x^i), define a map

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k, \quad v^i \frac{\partial}{\partial x^i}\Big|_p \mapsto \left(p, (v^1, \dots, v^n)\right).$$

This is linear on the fibers and satisfies $\pi_U \circ \Phi = \pi$. The composite map

$$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^k \xrightarrow{\varphi \times \mathrm{Id}_{\mathbb{R}^k}} \varphi(U) \times \mathbb{R}^k$$

is equal to the coordinate map $\tilde{\varphi} : \pi^{-1}(U) \longrightarrow \varphi(U) \times \mathbb{R}^k$ constructed in PROP. 3.12, so it is a diffeomorphism (as a composition of diffeomorphisms). Thus, Φ satisfies all the conditions for a smooth local trivialization. ∎

→ see ES11E1 for the uniqueness of the smooth structure on TM.

Any bundle that is not trivial requires more than one local trivialization. The next lemma shows that the composition of two smooth local trivializations has a simple form where they overlap.

· <u>LEM. 6.6</u>: Let $\pi : E \to M$ be a smooth vector bundle of rk k over M. Suppose that

$$\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k \quad \text{and} \quad \Psi : \pi^{-1}(V) \longrightarrow V \times \mathbb{R}^k$$

are two smooth local trivializations of E with $U \cap V \neq \emptyset$. Then there exists a smooth map

$$\tau : U \cap V \longrightarrow GL(k, \mathbb{R}),$$

called <u>the transition fnct between the smooth local trivializations Φ and Ψ</u>, such that the composition

$$\Phi \circ \Psi^{-1} : U \cap V \times \mathbb{R}^k \longrightarrow U \cap V \times \mathbb{R}^k$$

has the form

$$\Phi \circ \Psi^{-1}(p, v) = (p, \underset{\underset{GL(n,\mathbb{R})}{\cap}}{\tau(p)} \underset{\underset{\mathbb{R}^k}{\cap}}{v}).$$

<u>PROOF</u>: Note that the following diagram commutes:

$$\begin{tikzcd} U\cap V\times\mathbb{R}^k \arrow[dr, "\pi_1"'] & \pi^{-1}(U\cap V) \arrow[l, "\Psi"'] \arrow[r, "\Phi"] \arrow[d, "\pi"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & U\cap V\times\mathbb{R}^k \arrow[dl, "\pi_1=\pi_{U\cap V}"] \\ & U\cap V & \end{tikzcd}$$

and thus $\pi_1\circ(\Phi\circ\Psi^{-1})=\pi_1$, which means that

$$(\Phi\circ\Psi^{-1})(p,v)=(p,\sigma(p,v))$$

for some smooth map $\sigma: U\cap V\times\mathbb{R}^k\longrightarrow\mathbb{R}^k$ $\big((p,v)\in U\cap V\times\mathbb{R}^k \rightsquigarrow$
$\rightsquigarrow (\Phi\circ\Psi^{-1})(p,v)=(q,w)\in U\cap V\times\mathbb{R}^k \xrightarrow[\text{tivity}]{\text{Commuta-}} q=p$, and $w=w(p,v)=\sigma(p,v)\big)$.

Moreover, for each fixed $p\in U\cap V$, the map $v\mapsto\sigma(p,v)$ is an invertible linear map (since both $\Phi|_{E_p}$ and $\Psi|_{E_p}$ are $\mathbb{R}$-linear isomorphisms), so there is an invertible $k\times k$ matrix $\tau(p)$ s.t. $\sigma(p,v)=\tau(p)\cdot v$. It remains t.s.t. $\tau: U\cap V\longrightarrow GL(k,\mathbb{R})$ is smooth; this is ES10E1(b). ∎

Vector bundles are often most easily described by giving a collection of vector spaces, one for each pt of the base mnfd. In order to make such a set into a vector bundle, we would first have to construct a mnfd topology and a smooth structure on the disjoint union of all the vector spaces, and then construct the local trivializations and show that they have the requisite properties. The next lemma provides a shortcut (cf. LEM. 1.9) by showing that it is sufficient to construct the local trivializations, as long as they overlap with smooth transition fncts. (see also ES10E2 for a stronger form of this result.)

· <u>LEM 6.7</u> (vector bundle chart lemma): Let M be a smooth mnfd. Suppose that for each $p \in M$ we are given an $\mathbb{R}$-v.s. E_p of some fixed dim k. Set $E := \bigsqcup_{p \in M} E_p$ and consider the map $\pi : E \to M$, $v \in E_p \mapsto p \in M$. Suppose furthermore that we are given the following data:

(i) an open cover $\{U_\alpha\}_{\alpha \in A}$ of M,

(ii) for each $\alpha \in A$, a bijective map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is an $\mathbb{R}$-v.s. isomorphism from E_p to $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$,

(iii) for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ s.t. the map $\Phi_\alpha \circ \Phi_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^k \to U_\alpha \cap U_\beta \times \mathbb{R}^k$ has the form

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p) v).$$

Then E has a unique topology and smooth structure making it into a smooth mnfd and a smooth vector bundle of rk k over M with π as projection and $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in A}$ as smooth local trivializations.

$\rightsquigarrow$ proof: [Lee, Lemma 10.6]

$\rightsquigarrow$ application: [Lee, Example 10.7]

· REM. 6.8 (Restriction of a vector bundle):

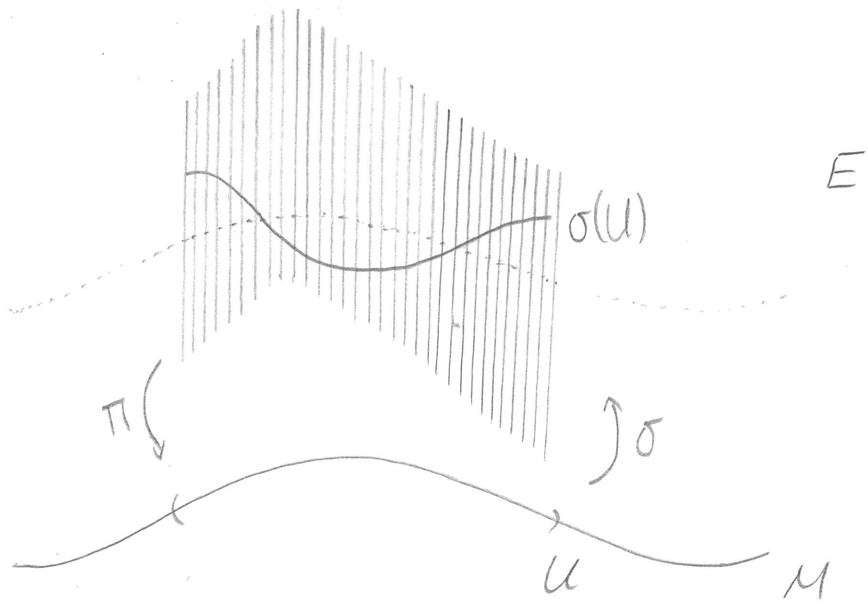
Let $\pi: E \to M$ be a rk k vector bundle and let $S \subseteq M$ be any subset. We define the restriction of E to S to be the set $E|_S = \bigcup_{p \in S} E_p$ with the projection $E|_S \to S$ obtained by restricting π. If $\Phi: \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a local trivialization of E over $U \subseteq M$, it restricts to a bijective map from $(\pi|_S)^{-1}(U \cap S)$ to $(U \cap S) \times \mathbb{R}^k$, and it is easy to check that these form local trivializations for a vector bundle structure on $E|_S$.

If E is a smooth vector bundle over M and $S \subseteq M$ is an embedded submfd, it follows easily from LEM 6.7 that $E|_S$ is a smooth vector bundle.

Finally, if E is a smooth vector bundle over M, but $S \subseteq M$ is merely immersed, then we give $E|_S$ a topology and a smooth structure making it into a smooth rk k vector bundle over S as follows: For each $p \in S$ choose a neighborhood U of p in M over which there is a smooth local trivialization Φ of E, and a neighborhood V of p in S that is embedded in M and contained in U. Then the restriction of Φ to $\pi^{-1}(V)$ is a bijection from $\pi^{-1}(V)$ to $V \times \mathbb{R}^k$, and we can apply LEM 6.7 to these bijections to yield the desired structure.

In particular, if $S \subseteq M$ is a smooth (immersed or embedded) submfd, then $TM|_S$ is called the ambient tangent bundle over S.

DEF. 6.9: Let $\pi: E \to M$ be a vector bundle. A <u>local section of E</u> is a cont. map $\sigma: U \to E$ defined on some open subset $U \subseteq M$ and satisfying $\pi \circ \sigma = Id_U$.



This means that $\sigma(p) \in E_p$ for every $p \in U$. A <u>global section of E</u> is a section of E defined on all of M, i.e., a cont. map $\sigma: M \to E$ s.t. $\pi \circ \sigma = Id_M$.

A <u>rough (local or global) section of E</u> over an open subset $U \subseteq M$ is defined to be a (not necessarily continuous) map $\sigma: U \to M$ s.t. $\pi \circ \sigma = Id_U$. (Note that a local section of E over U is the same as a global section of the restricted bundle $E|_U$.)

<u>The zero section of E</u> is the global section $\zeta: M \to E$ of E defined by $\zeta(p) = 0 \in E_p$ for each $p \in M$.

If M is a smooth mnfd and if E is a smooth vector bundle over M, then a <u>smooth (local or global) section</u> of E is one that is a smooth map from its domain to E.

$\rightsquigarrow \zeta$ is cont / smooth : ES10E3(a)

$\rightsquigarrow$ sections of the product bundle : ES10E3(c)

If $E \to M$ is a smooth vector bundle, then the set of all smooth global sections of E is an $\mathbb{R}$-v.s. under pointwise addition and scalar multiplication:

$$(c_1\sigma_1 + c_2\sigma_2)(p) := c_1\sigma_1(p) + c_2\sigma_2(p)$$

This vector space is usually denoted by $\Gamma(E)$ (but for particular vector bundles we often introduce specialized notation for their spaces of global sections).

Smooth sections of $E \to M$ can be multiplied by smooth real-valued fncts:

$$f \in C^\infty(M),\ \sigma \in \Gamma(E) \rightsquigarrow f\sigma \in \Gamma(E),\ (f\sigma)(p) := \underbrace{f(p)}_{\in\mathbb{R}}\underbrace{\sigma(p)}_{\in E_p}.$$

$\rightsquigarrow$ the various claims made above will be proved in ES10E3.

· <u>LEM 6.10</u> (Extension lemma for vector bundles): Let $\pi: E \to M$ be a smooth vector bundle. Let $A \subseteq M$ be a closed subset, and let $\sigma: A \to E$ be a section of $E|_A$ that is smooth in the sense that σ extends to a smooth local section of E in a neighborhood of each pt. Then for each open subset $U \subseteq M$ containing A, there exists a smooth global section $\tilde{\sigma} \in \Gamma(E)$ s.t. $\tilde{\sigma}|_A = \sigma$ and $\operatorname{supp}\tilde{\sigma}\ \left(= \overline{\{p \in M \mid \tilde{\sigma}(p) \neq 0\}}\right) \subseteq U$.

<u>PROOF</u>: Exercise! (Similar to the proof of LEM 9.15) ■

$\rightsquigarrow$ see ES10E3(d) and ES10E4(d) for applications

__DEF. 6.11__: Let $E \to M$ be a vector bundle. If $U \subseteq M$ is an open subset, then a k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ of E over U is said to be <u>linearly independent</u> if their values $(\sigma_1(p), \dots, \sigma_k(p))$ form a linearly independent k-tuple in E_p for each $p \in U$. Similarly, they are said to <u>span</u> E if their values span E_p for each $p \in U$.

A <u>local frame for E over U</u> is an ordered k-tuple $(\sigma_1, \dots, \sigma_k)$ of linearly independent local sections of E over U that span E; thus, $(\sigma_1(p), \dots, \sigma_k(p))$ is a basis for E_p for each $p \in U$. It is called a <u>global frame</u> if $U = M$.

If, moreover, $E \to M$ is a smooth vector bundle, then a <u>local or global frame</u> for E is said to be <u>smooth</u> if each σ_i is a smooth section of E. (We often denote a frame $(\sigma_1, \dots, \sigma_k)$ by (σ_i).)

· __EXAMPLE 6.12__ (Global frame for a product bundle): If $E := M \times \mathbb{R}^k \to M$ is a (smooth) product bundle over a (smooth) mnfd M, then the standard basis $(e_1, \dots, e_k)$ for $\mathbb{R}^k$ yields a (smooth) global frame $\tilde{e}_i$ for E, defined by

$$\tilde{e}_i : M \to E, \ p \mapsto (p, e_i).$$

$\rightsquigarrow$ For the correspondence between smooth local frames and smooth local trivializations see ES10E5 (which also settles the question of the existence of smooth local frames). See also ES11E1 (uniqueness of smooth structure on TM)

$\rightsquigarrow$ For the completion of smooth local frames of smooth vector bundles see ES10E4.

We conclude this chapter with the important observation that smoothness of sections of vector bundles can be characterized in terms of local frames:

Assume that (σ_i) is a smooth local frame for E over some open subset $U \subseteq M$. If $\tau: M \to E$ is a rough section, the value of τ at an arbitrary pt $p \in U$ can be written as

$$\tau(p) = \tau^i(p)\,\sigma_i(p)$$

for some uniquely determined numbers $(\tau^1(p), \dots, \tau^k(p))$. This defines k fncts $\tau^i: U \to \mathbb{R}$, called <u>the component fncts of τ w.r.t. the given local frame (σ_i)</u>.

· <u>PROP. 6.13</u> (Local frame criterion for cont./smoothness): Let $\pi: E \to M$ be a cont. (resp. smooth) vector bundle and let $\tau: M \to E$ be a rough section. If (σ_i) is a cont. (resp. smooth) local frame for E over an open subset $U \subseteq M$, then τ is cont. (resp. smooth) iff its component fncts w.r.t. (σ_i) are cont. (resp. smooth).

<u>PROOF</u>: We prove the statement in the smooth case; the other case can be treated similarly. Let $\Phi: \pi^{-1}(U) \to U \times \mathbb{R}^k$ be the smooth local trivialization associated with the smooth local frame; see ES10E5(a)(b). Since Φ is a diffeomorphism, τ is smooth on U iff $\Phi \circ \tau$ is smooth on U. By the construction

n ES10E5(b) we know that

$$(\Phi \circ \tau)(p) = \left(p, (\tau^1(p), \dots, \tau^k(p))\right),$$

where (τ^i) are the component fncts of τ w.r.t. (σ_i), so $\Phi \circ \tau$ is smooth iff the component fncts τ^i are smooth according to ES3E4(b).

Note that PROP. 6.13 applies equally well to local sections, since a local section of E over an open subset $V \subseteq M$ is a global section of the restricted bundle $E|_V$.