Midterm Exam

Exercice 1. (21 + 3 points)

On the state space $S = \{0,1\}^d$, with $d \geq 2$ integer, consider the time-homogeneous Markov chain $(X_n,\, n \geq 0)$ with the following behaviour:

At each time step $n \geq 0$, choose a number $i \in \{1, \ldots, d\}$ uniformly at random, and also independently of the numbers chosen at previous time steps; then assuming that $X_n = x$, set

$$(X_{n+1})_i = \begin{cases} 0 & \text{with probability } \frac{|x|}{d} \\ 1 & \text{with probability } 1 - \frac{|x|}{d} \end{cases} \tag{1}$$

as well as $(X_{n+1})_j = x_j$ for every $j \neq i$.

NB: Recall that $|x|$ stands for the number of 1's in the vector x.

a) Explain why the process $(X_n,\, n \geq 0)$ is a Markov chain (a *short* justification will do here).

b) Do the states $x_0 = (0, 0, \ldots, 0)$ and $x_1 = (1, 1, \ldots, 1)$ communicate in this chain? Is the chain irreducible? Justify.

c) Is the chain aperiodic for any value of $d \geq 2$? Justify.

From now on, we restrict ourselves to the particular case $d = 3$.

d) Write down explicitly the transition matrix P of the chain (it might perhaps be a good idea to also draw the transition graph, but this is not required here).

e) Compute the (unique) stationary distribution π of the chain. Does detailed balance hold?

f) Is π also a limiting distribution? Justify.

BONUS g) Explain what major change(s) would happen in the Markov chain $(X_n, n \geq 0)$ if we inverted the roles of 0 and 1 in equation (1).

Exercice 2. (17 points)

Let $(X_n,\, n \geq 0)$ be a Markov chain with state space $S = \{0,1,2,3\}$ and transition matrix P given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & b & b & 0 \\ 0 & b & b & a \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where the parameters $0 \leq a \leq 1$ and $0 \leq b \leq \frac{1}{2}$ are such that $a + 2b = 1$.

a) For what values of a is the chain $(X_n,\, n \geq 0)$ ergodic? Justify.

b) Compute the unique limiting and stationary distribution π of the chain in this case.

c) Compute the eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3$ of the matrix P.

Hint: Look for eigenvectors of the form (u, v, v, u) and $(u, v, -v, -u)$.

d) Express the spectral gap γ of the chain as a function of the parameter a.

e) Is there a way to increase the spectral gap γ by making the chain lazy (i.e., adding self-loops of equal weight to each state)? Justify.

Exercise 3. (12 points)

Let $(X_n,\, n \in \mathbb{N})$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = \mathbb{P}(X_n = 2) = \frac{1}{3}$ for every $n \geq 0$.

a) Let $(Y_n,\, n \in \mathbb{N})$ be the process defined as

$$Y_n = \prod_{j=0}^{n} X_j, \quad n \geq 0$$

Explain why $(Y_n,\, n \in \mathbb{N})$ is a Markov chain. Is it time-homogeneous?

b) What is the state space S of the chain $(Y_n,\, n \in \mathbb{N})$? And which states are recurrent/transient?

c) Let now $(Z_n,\, n \in \mathbb{N})$ be the process defined as

$$Z_n = \max_{0 \leq j \leq n} Y_j, \quad n \geq 0$$

Is $(Z_n,\, n \in \mathbb{N})$ also a Markov chain? Is it time-homogeneous? Justify.