**Midterm Solutions**

**Exercice 1. (21 + 3 points)**

On the state space $S = \{0,1\}^d$, with $d \geq 2$ integer, consider the time-homogeneous Markov chain $(X_n,\, n \geq 0)$ with the following behaviour:

At each time step $n \geq 0$, choose a number $i \in \{1, \ldots, d\}$ uniformly at random, and also independently of the numbers chosen at previous time steps; then assuming that $X_n = x$, set

$$(X_{n+1})_i = \begin{cases} 0 & \text{with probability } \frac{|x|}{d} \\ 1 & \text{with probability } 1 - \frac{|x|}{d} \end{cases} \tag{1}$$

as well as $(X_{n+1})_j = x_j$ for every $j \neq i$.

*NB:* Recall that $|x|$ stands for the number of 1's in the vector $x$.

**a)** Explain why the process $(X_n,\, n \geq 0)$ is a Markov chain (a *short* justification will do here).

**Answer:** In short, the steps performed by the process are indepedendent from each other, so the Markov property holds. In more detail, we have

$$\mathbb{P}(X_{n+1} = y | X_n = x, X_{n-1} = x_{n-1}, \cdots, X_0 = x_0) = \\ \begin{cases} \frac{1}{d}\left(1 - \frac{|x|}{d}\right)^{y_i}\left(\frac{|x|}{d}\right)^{1-y_i} & \text{if } x \text{ and } y \text{ differ only at the } i^{\text{th}} \text{ coordinate} \\ \frac{1}{d}\sum_{i=1}^{d}\left(1 - \frac{|x|}{d}\right)^{y_i}\left(\frac{|x|}{d}\right)^{1-y_i} & \text{if } y = x \\ 0 & \text{otherwise,} \end{cases}$$

which depends only on $x$ and $y$, and hence $(X_n,\, n \geq 0)$ is a Markov chain.

**b)** Do the states $x_0 = (0, 0, \ldots, 0)$ and $x_1 = (1, 1, \ldots, 1)$ communicate in this chain? Is the chain irreducible? Justify.

**Answer:** Yes, $x_0$ and $x_1$ communicate. The Markov chain is irreducible since any two states communicate with each other.

**c)** Is the chain aperiodic for any value of $d \geq 2$? Justify.

**Answer:** The chain is aperiodic for $d \geq 2$. This is since all the states except $x_0 = (0, 0, \ldots, 0)$ and $x_1 = (1, 1, \ldots, 1)$ have self-loops.

*From now on, we restrict ourselves to the particular case $d = 3$.*

**d)** Write down explicitly the transition matrix $P$ of the chain (it might perhaps be a good idea to also draw the transition graph, but this is not required here).

**Answer:** Considering the order $(000),(001),(010),(011),(100),(101),(110),(111)$ for the states of $S$, we obtain

$$P = \begin{pmatrix} 0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \\ 1/9 & 4/9 & 0 & 2/9 & 0 & 2/9 & 0 & 0 \\ 1/9 & 0 & 4/9 & 2/9 & 0 & 0 & 2/9 & 0 \\ 0 & 2/9 & 2/9 & 4/9 & 0 & 0 & 0 & 1/9 \\ 1/9 & 0 & 0 & 0 & 4/9 & 2/9 & 2/9 & 0 \\ 0 & 2/9 & 0 & 0 & 2/9 & 4/9 & 0 & 1/9 \\ 0 & 0 & 2/9 & 0 & 2/9 & 0 & 4/9 & 1/9 \\ 0 & 0 & 0 & 1/3 & 0 & 1/3 & 1/3 & 0 \end{pmatrix}$$

**e)** Compute the (unique) stationary distribution $\pi$ of the chain. Does detailed balance hold?

**Answer:** Let's see if there exists a $\pi$ that satisfies the detailed balance equation. We have

$$\pi_2 = \frac{p_{12}}{p_{21}}\pi_1 \implies \pi_2 = 3\pi_1.$$

Similarly, we get

$$\begin{aligned} \pi_3 &= 3\pi_1, \\ \pi_4 &= \pi_2 = 3\pi_1, \\ \pi_5 &= 3\pi_1, \\ \pi_6 &= \pi_2 = 3\pi_1, \\ \pi_7 &= \pi_3 = 3\pi_1, \\ \pi_8 &= \pi_4/3 = \pi_1. \end{aligned}$$

Imposing $\sum_i \pi_i = 1$, we get $\pi_1 = \frac{1}{20}$. Hence, $\pi = \frac{1}{20}(1,3,3,3,3,3,3,1)$ and the chain satisfies detailed balance.

**f)** Is $\pi$ also a limiting distribution? Justify.

**Answer:** The given Markov chain is irreducible, positive-recurrent (due to finite state space), and aperiodic, therefore ergodic. Hence $\pi$ is also the limiting distribution.

**BONUS g)** Explain what major change(s) would happen in the Markov chain $(X_n, n \geq 0)$ if we inverted the roles of 0 and 1 in equation (1).

**Answer:** There would be 3 equivalence classes: $\{x_0 = (0,0,\ldots,0)\}$, and $\{x_1 = (1,1,\ldots,1)\}$ that are recurrent and $\{0,1\}^d \backslash \{x_0, x_1\}$ which is transient.

**Exercice 2. (17 points)**

Let $(X_n,\, n \geq 0)$ be a Markov chain with state space $S = \{0,1,2,3\}$ and transition matrix $P$ given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & b & b & 0 \\ 0 & b & b & a \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where the parameters $0 \leq a \leq 1$ and $0 \leq b \leq \frac{1}{2}$ are such that $a + 2b = 1$.

**a)** For what values of $a$ is the chain $(X_n,\, n \geq 0)$ ergodic? Justify.

**Answer:** The chain is ergodic for $a \in (0,1)$. For $a \in \{0,1\}$, the chain is reducible.

**b)** Compute the unique limiting and stationary distribution $\pi$ of the chain in this case.

**Answer:** Solving $\pi = \pi P$ and using $a + 2b = 1$, we get $\pi = \frac{1}{2(1+a)}\,(a,1,1,a)$.

**c)** Compute the eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3$ of the matrix $P$.

*Hint:* Look for eigenvectors of the form $(u,v,v,u)$ and $(u,v,-v,-u)$.

**Answer:** We have

$$P \begin{pmatrix} u \\ v \\ v \\ u \end{pmatrix} = \begin{pmatrix} v \\ au + 2bv \\ au + 2bv \\ v \end{pmatrix} = \frac{v}{u} \begin{pmatrix} u \\ u(au/v + 2b) \\ u(au/v + 2b) \\ u \end{pmatrix}.$$

Hence $(u,v,v,u)^T$ is an eigenvector for $u,v$ such that $\frac{u}{v}(a\frac{u}{v} + 2b) = 1$ which is satisfied for $u = v$ and $v = -au$. Corresponding eigenvalues are 1 and $-a$. Also, we have

$$P \begin{pmatrix} u \\ v \\ -v \\ -u \end{pmatrix} = \begin{pmatrix} v \\ au \\ -au \\ -v \end{pmatrix} = \frac{v}{u} \begin{pmatrix} u \\ au^2/v \\ -au^2/v \\ -u \end{pmatrix}.$$

Hence $[u,v,-v,-u]^T$ is an eigenvector for $u,v$ such that $\frac{au^2}{v} = v$ which is satisfied for $v = \pm\sqrt{a}u$. Corresponding eigenvalues are $\pm\sqrt{a}$. Therefore the eigenvalues of $P$ are $1 \geq \sqrt{a} \geq -a \geq -\sqrt{a}$.

**d)** Express the spectral gap $\gamma$ of the chain as a function of the parameter $a$.

**Answer:** Since $a \in [0,1]$, we have $\lambda_* = \max\{\lambda_1, -\lambda_3\} = \sqrt{a}$. Hence the spectral gap $\gamma = 1 - \sqrt{a}$.

**e)** Is there a way to increase the spectral gap $\gamma$ by making the chain lazy (i.e., adding self-loops of equal weight to each state)? Justify.

**Answer:** Let us add self loops of strength $\alpha \in (0,1)$ to each state. Then, the new transition probability matrix is $\alpha I + (1-\alpha)P$ whose eigenvalues are $1$, $\alpha + (1-\alpha)\sqrt{a}$, $\alpha + (1-\alpha)a$, $\alpha - (1-\alpha)\sqrt{a}$. The eigenvalue $\alpha + (1-\alpha)\sqrt{a}$ is between $\sqrt{a}$ and 1 for any $\alpha$. Hence we can only reduce the spectral gap.

**Exercise 3. (12 points)**

Let $(X_n,\, n \in \mathbb{N})$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = \mathbb{P}(X_n = 2) = \frac{1}{3}$ for every $n \geq 0$.

**a)** Let $(Y_n,\, n \in \mathbb{N})$ be the process defined as

$$Y_n = \prod_{j=0}^{n} X_j, \quad n \geq 0$$

Explain why $(Y_n,\, n \in \mathbb{N})$ is a Markov chain. Is it time-homogeneous?

**Answer:** We have

$$\begin{aligned}
\mathbb{P}\left(Y_{n+1} = j | Y_n = i, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0\right) &= \mathbb{P}\left(\prod_{l=0}^{n+1} X_l = j \,\middle|\, \prod_{l=0}^{n} X_l = i, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0\right) \\
&= \mathbb{P}\left(X_{n+1} = j/i \,\middle|\, \prod_{l=0}^{n} X_l = i, Y_{n-1} = y_{n-1}, \ldots, Y_0 = y_0\right) \\
&= \mathbb{P}\left(X_{n+1} = j/i\right)
\end{aligned}$$

Therefore $(Y_n,\, n \in \mathbb{N})$ is a time-homogenous Markov chain.

**b)** What is the state space $S$ of the chain $(Y_n,\, n \in \mathbb{N})$? And which states are recurrent/transient?

**Answer:** The state space of $(Y_n,\, n \in \mathbb{N})$ is $S = \{2^n : n \in \mathbb{N}\} \cup \{0\}$. State 0 is recurrent. All other states are transient.

**c)** Let now $(Z_n,\, n \in \mathbb{N})$ be the process defined as

$$Z_n = \max_{0 \leq j \leq n} Y_j, \quad n \geq 0$$

Is $(Z_n,\, n \in \mathbb{N})$ also a Markov chain? Is it time-homogeneous? Justify.

**Answer:** $(Z_n,\, n \in \mathbb{N})$ is not a Markov chain. Consider the sample paths that can make $Z_1 = 2$. They are $(X_0, X_1) \in \{(1,2), (2,1), (2,0)\}$. The corresponding values of $(Z_0, Z_1)$ are $(1,2), (2,2), (2,2)$. Hence we have

$$\mathbb{P}(Z_2 = 2 | Z_1 = 2) = 7/9 \qquad \text{and} \qquad \mathbb{P}(Z_2 = 4 | Z_1 = 2) = 2/9,$$

while

$$\mathbb{P}(Z_2 = 2 | Z_1 = 2, Z_0 = 1) = 2/3 \qquad \text{and} \qquad \mathbb{P}(Z_2 = 4 | Z_1 = 2, Z_0 = 1) = 1/3.$$

Therefore, $(Z_n,\, n \in \mathbb{N})$ is not a Markov chain (and therefore neither time-homogeneous).