Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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# Exercise Sheet 11

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**Exercise 1** (*Uniqueness of the smooth structure on $TM$*)**:**
Let $M$ be a smooth $n$-manifold. Show that the topology and smooth structure on the tangent bundle $TM$ constructed in *Proposition 3.12* are the unique ones with respect to which $\pi\colon TM \to M$ is a smooth vector bundle with the given vector space structure on the fibers, and such that all coordinate vector fields are smooth local sections.

[Hint: Use *Exercise* 5*, Sheet* 10.]

**Exercise 2 (to be submitted by Friday, 08.12.2023, 20:00):**

(a) Consider the tangent bundle $\pi\colon T\mathbb{S}^2 \to \mathbb{S}^2$ of the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Compute the transition function associated with the two local trivializations determined by stereographic coordinates.

(b) Show that there is a smooth vector field on $\mathbb{S}^2$ which vanishes at exactly one point.

[Hint: Use the stereographic projection and consider one of the coordinate vector fields.]

**Exercise 3:**
Consider the *Euler vector field* on $\mathbb{R}^n$, i.e., the vector field $V$ on $\mathbb{R}^n$ whose value at a point $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ is

$$V_x = x^1 \frac{\partial}{\partial x_1}\bigg|_x + \dots + x^n \frac{\partial}{\partial x_n}\bigg|_x .$$

(a) Check that $V$ is a smooth vector field on $\mathbb{R}^n$.

(b) Let $c \in \mathbb{R}$ and let $f\colon \mathbb{R}^n \smallsetminus \{0\} \to \mathbb{R}$ be a smooth function which is *positively homogeneous of degree* $c$, i.e., $f(\lambda x) = \lambda^c f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Prove that $Vf = cf$.

[Hint: Differentiate the above relation with respect to both $x^i$ and $\lambda$.]

(c) Compute the integral curves of $V$.

**Definition.** Let $F\colon M \to N$ be a smooth map between smooth manifolds and let $X$ be a vector field on $X$. If there exists a vector field $Y$ on $N$ such that $dF_p(X_p) = Y_{F(p)}$ for each $p \in M$, then $X$ and $Y$ are said to be *$F$-related*.[1]

**Exercise 4:**

(a) Let $F\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Show that $X$ and $Y$ are $F$-related if and only if for every smooth real-valued function $f$ defined on an open subset of $N$, we have

$$X(f \circ F) = (Yf) \circ F.$$

(b) Consider the smooth map

$$F\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\cos t, \sin t)$$

and the smooth vector fields

$$X = \frac{d}{dt} \in \mathfrak{X}(\mathbb{R}) \quad \text{and} \quad Y = x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2).$$

Show that $X$ and $Y$ are $F$-related.

(c) Let $F\colon M \to N$ be a diffeomorphism and let $X \in \mathfrak{X}(M)$. Prove that there exists a unique smooth vector field $Y$ on $N$ that is $F$-related to $X$. The vector field $Y$ is denoted by $F_*X$ and is called the *pushforward of $X$ by $F$*.

(d) Consider the open submanifolds

$$M := \big\{(x,y) \in \mathbb{R}^2 \mid y > 0 \ \text{ and } \ x + y > 0\big\}$$

and

$$N := \big\{(u,v) \in \mathbb{R}^2 \mid u > 0 \ \text{ and } \ v > 0\big\}$$

of $\mathbb{R}^2$ and the map

$$F\colon M \to N,\ (x,y) \mapsto \big(x + y, \frac{x}{y} + 1\big).$$

(i) Show that $F$ is a diffeomorpism and compute its inverse $F^{-1}$.

(ii) Compute the pushforward $F_*X$ of the following smooth vector field $X$ on $M$:

$$X_{(x,y)} = y^2 \frac{\partial}{\partial x}\bigg|_{(x,y)}.$$

(e) *Naturality of integral curves*: Let $F\colon M \to N$ be a smooth map. Show that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are $F$-related if and only if $F$ takes integral curves of $X$ to integral curves of $Y$.

[1] In general, if $F\colon M \to N$ is a smooth map and if $X$ is a (rough) vector field on $M$, then for each point $p \in M$ we obtain a tangent vector $dF_p(X_p) \in T_{F(p)}N$ by applying the differential of $F$ at $p$ to the tangent vector $X_p \in T_pM$. However, this does not define a vector field on $N$ in general. For example, if $F$ is not surjective, there is no way to decide what tangent vector to assign to a point $q \in N \setminus F(M)$, while if $F$ is not injective, then for some points of $N$ there may be several different vectors obtained by applying $dF$ to $X$ at different points of $M$.

**Exercise 5:**
Let $M$ be a smooth $n$-manifold and let $X$ and $Y$ be smooth vector fields on $M$.

(a) *Coordinate formula for the Lie bracket*: Let

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$$

be the coordinate expressions for $X$ and $Y$, respectively, in terms of some smooth local coordinates $(x^i)$ for $M$. Show that the Lie bracket $[X, Y]$ has the following coordinate expression:

$$[X, Y] = \sum_{j=1}^{n} \sum_{i=1}^{n} \left( X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

(b) Compute the Lie brackets $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$ of the coordinate vector fields $\partial/\partial x^i$ in any smooth chart $\big(U, (x^i)\big)$ for $M$.

(c) Assume now that

$$M = \mathbb{R}^3, \quad X = x\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\,\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial z},$$

and compute the Lie bracket $[X, Y]$.

**Exercise 6** (*Properties of the Lie bracket*)**:**
Let $M$ be a smooth manifold. Show that the Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:

(a) *Bilinearity*: For all $a, b \in \mathbb{R}$ we have

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(b) *Antisymmetry*:

$$[X, Y] = -[Y, X].$$

(c) *Jacobi identity*:

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$$

(d) For all $f, g \in C^\infty(M)$ we have

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

**Exercise 7:**
Let $F\colon M \to N$ be a smooth map.

(a) *Naturality of the Lie bracket*: Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be vector fields such that $X_i$ is $F$-related to $Y_i$ for $i \in \{1, 2\}$. Show that $[X_1, X_2]$ is $F$-related to $[Y_1, Y_2]$.

(b) *Pushforwards of Lie brackets*: Assume that $F$ is a diffeomorphism and consider $X_1, X_2 \in \mathfrak{X}(M)$. Show that

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2]\,.$$