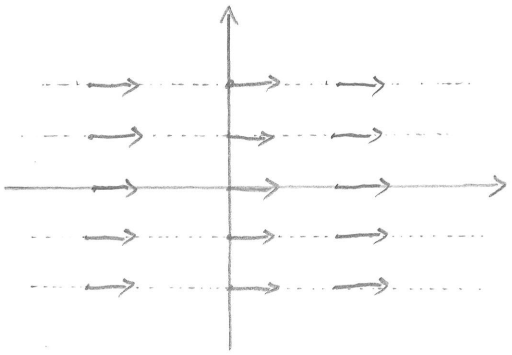
EXAMPLE 7.11: Let (x,y) be the standard coordinates on $\mathbb{R}^2$.

1) Consider $V = \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$. The integral curves of V are precisely the straight lines parallel to the x-axis, with parametrizations of the form $\gamma(t) = (a+t, b)$ for constants $a, b \in \mathbb{R}$.



Thus, there is a unique integral curve starting at each pt of the plane, and the images of different integral curves are either identical or disjoint. L11
L12

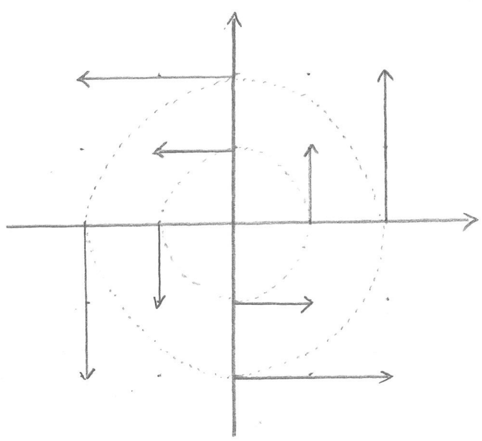
2) Consider $W = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$. To determine the integral curves of W we proceed as follows (see p. 97):

$$\gamma(t) = (\gamma^1(t), \gamma^2(t)) \rightsquigarrow \dot{\gamma}(t) = W_{\gamma(t)} \Rightarrow$$

$$\Rightarrow \begin{cases} \dot{\gamma}_1(t) = -\gamma_2(t) \\ \dot{\gamma}_2(t) = \gamma_1(t) \end{cases} \xrightarrow{\ddot{\gamma}_1(t) + \gamma_1(t) = 0}$$

$$\Rightarrow \begin{cases} \gamma_1(t) = a\cos t - b\sin t \\ \gamma_2(t) = a\sin t + b\cos t \quad (= -\dot{\gamma}_1(t)) \end{cases}$$

for constants $a, b \in \mathbb{R}$. Thus, each curve of the form



$$\gamma(t) = (a\cos t - b\sin t, a\sin t + b\cos t), \ t \in \mathbb{R}$$

is an integral curve of W. When $(a,b) = (0,0)$, this is the constant curve $\gamma(t) \equiv (0,0)$, otherwise, it is a circle traversed clockwise. Since $\gamma(0) = (a,b)$, we see again that there is a unique integral curve starting at each pt $(a,b) \in \mathbb{R}^2$, and the images of the various integral curves are either identical or disjoint.

· <u>DEF. 7.12</u>: Let M be a smooth mnfd.

(a) A <u>flow domain</u> for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set

$$\mathcal{D}^{(p)} := \{ t \in \mathbb{R} \mid (t,p) \in \mathcal{D} \} \subseteq \mathbb{R}$$

is an open interval containing $0 \in \mathbb{R}$.

(b) A <u>flow</u> on M is a cont. map $\theta : \mathcal{D} \to M$, where $\mathcal{D} \subseteq$ $\subseteq \mathbb{R} \times M$ is a flow domain, which satisfies the following group laws:

· $\forall p \in M$: $\theta(0,p) = p$

· $\forall s \in \mathcal{D}^{(p)}$ $\forall t \in \mathcal{D}^{(\theta(s,p))}$: $s+t \in \mathcal{D}^{(p)}$ we have

$$\theta(t, \theta(s,p)) = \theta(t+s, p).$$

When $\mathcal{D} = \mathbb{R} \times M$ (and hence θ is a cont. left $\mathbb{R}$-action on M) we say that θ is a <u>global flow</u> on M (or a <u>one-parameter group action</u>).

(c) A <u>maximal flow</u> on M is a flow that admits no extension to a flow on a larger flow domain.

Let $\theta : \mathcal{D} \to M$ be a flow on M.

· For each $p \in M$ we define a map

$$\theta^{(p)} : \mathcal{D}^{(p)} \longrightarrow M, \quad \theta^{(p)}(t) = \theta(t,p).$$

· For each $t \in \mathbb{R}$ we define a set

$$M_t := \{p \in M \mid (t,p) \in \mathcal{D}\}$$

and a map

$$\theta_t : M_t \longrightarrow M, \quad \theta_t(p) := \theta(t,p).$$

These maps satisfy $\theta_t \circ \theta_s = \theta_{t+s}$ and $\theta_0 = Id_M$, so each θ_t is a homeomorphism, and if θ is smooth, then each θ_t is in fact a diffeomorphism.

Note that $p \in M_t \Leftrightarrow (t,p) \in \mathcal{D} \Leftrightarrow t \in \mathcal{D}^{(p)}$.

· PROP. 7.13: If $\theta : \mathcal{D} \rightarrow M$ is a smooth flow on M, then the infinitesimal generator V of θ, defined as

$$V : M \longrightarrow TM, \quad p \longmapsto V_p := \theta^{(p)\prime}(0) = \left.\frac{d}{dt}\right|_{t=0} \theta^{(p)}(t),$$

is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve of V starting at $p \in M$.

PROOF: When $\mathcal{D} = \mathbb{R} \times M$, this is ES12E4. The proof of the general case is essentially identical to the proof for global flows. ∎

· THM 7.14 (Fundamental theorem on flows): Let V be a smooth vector field on a smooth mnfd M. There is a unique smooth maximal flow $\theta : \mathcal{D} \rightarrow M$ whose infinitesimal generator is V. This flow has the following properties:

(a) For each $p \in M$, the curve $\theta^{(p)} : \mathcal{D}^{(p)} \rightarrow M$ is the unique maximal integral curve of V (maximal in the sense that it cannot be extended to an integral curve on any larger open interval) starting at p.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval
$$\mathcal{D}^{(\theta(s,p))} = \mathcal{D}^{(p)} - s = \{ t - s \mid t \in \mathcal{D}^{(p)} \}.$$
(c) For each $t \in \mathbb{R}$, the set M_t is open in M, and the map $\theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

⤳ For the proof we refer to [Lee, Thm 9.12]

⤳ We have $p \in M_t \Rightarrow t \in \mathcal{D}^{(p)} \overset{(b)}{\Rightarrow} \mathcal{D}^{(\theta(t,p))} = \mathcal{D}^{(p)} - t$
$$\overset{\text{dfn}}{\Rightarrow} -t \in \mathcal{D}^{(\theta(t,p))} \Rightarrow \theta_t(p) = \theta(t,p) \in M_{-t},$$
that is, $\theta_t : M_t \to M_{-t}$, $t \in \mathbb{R}$.

The flow whose existence and uniqueness are asserted in THM 7.14 is called <u>the flow generated by V</u>, or just <u>the flow of V</u>.

· <u>EXAMPLE 7.15</u>: The two vector fields on $\mathbb{R}^2$ described in Ex. 7.11 had integral curves defined for all $t \in \mathbb{R}$, so they generate global flows. We can write down their flows explicitly:

- $\theta_V : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $(t,(x,y)) \mapsto (x+t, y)$

  (For each $t \in \mathbb{R} \setminus \{0\}$, $(\theta_V)_t$ translates the plane to the left ($t<0$) or to the right ($t>0$) by a distance $|t|$.)

- $\theta_W : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, $(t,(x,y)) \mapsto (x\cos t - y\sin t, x\sin t + y\cos t)$

  (For each $t \in \mathbb{R}$, $(\theta_W)_t$ rotates the plane through an angle t about the origin.)

However, there are also smooth vector fields whose integral curves are not defined for all $t \in \mathbb{R}$. Here are two examples:

1) $M := \mathbb{R}^2 \setminus \{(0,0)\}$, $V := \frac{\partial}{\partial x} \in \mathfrak{X}(M)$.

The unique integral curve of V starting at $(-1,0) \in M$ is $\gamma(t) = (t-1, 0)$. However, it cannot be extended continuously past $t=1$. (This is intuitely evident because of the "hole" in M at the origin)

2) $M := \mathbb{R}^2$, $W := x^2 \frac{\partial}{\partial x} \in \mathfrak{X}(M)$.

The unique integral curve of W starting at $(1,0)$ is $\gamma(t) = \left(\frac{1}{1-t}, 0\right)$. It cannot be extended past $t=1$, because its x-coordinate is unbounded as $t \nearrow 1$.

· <u>DEF. 7.16</u>: A smooth vector field V on a smooth mnfd M is called <u>complete</u> if it generates a global flow, or equivalently, if each of its maximal integral curves is defined for all $t \in \mathbb{R}$.

It is not always easy to determine by looking at a vector field whether it is complete or not. If one can solve the ODE explicitly to find all of the integral curves, and they all exist for all time, then the vector field is complete. On the other hand, if one can find one single integral curve that cannot be extended to all of $\mathbb{R}$, then it is not complete. However, it is often impossible to solve the ODE explicitly, so it is useful to have some general criteria for determining when a vector field is complete. The following theorem provides such a criterion. For its proof we refer to [Lee, Thm 9.16].

· THM 7.17: Every compactly supported smooth vector field on a smooth mnfd is complete.

In particular, on a compact smooth mnfd, every smooth vector field is complete.

CH. 8 : DIFFERENTIAL FORMS

· DEF. 8.1 : Let M be a smooth mnfd. For each $p \in M$ we define the cotangent space at p, denoted by T_p^*M, to be the dual space to T_pM:

$$T_p^*M := (T_pM)^*.$$

Elements of T_p^*M are called (tangent) covectors at $p \in M$.

Given smooth local coordinates (x^i) on an open subset $U \subseteq M$, for each $p \in U$ the coordinate basis $(\partial/\partial x^i|_p)$ for T_pM gives rise to a dual basis for T_p^*M, which we denote temporarily by $(\lambda^i|_p)$. Any covector $\omega \in T_p^*M$ can thus be written uniquely as

$$\omega = \omega_i \, \lambda^i|_p ,$$

where

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\Big|_p \right).$$

Given now another set of smooth local coordinates $(\tilde{x}^j)$ whose domain contains $p \in U$, denote by $(\tilde{\lambda}^j|_p)$ the basis for T_p^*M dual to $(\partial/\partial \tilde{x}^j|_p)$. We can compute the components of the same covector $\omega \in T_p^*M$ with respect to the new coordinate system as follows. Recall first that the coordinate vector fields transform as follows:

$$\frac{\partial}{\partial x^i}\Big|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(p) \frac{\partial}{\partial \tilde{x}^j}\Big|_p \qquad (*_1)$$

see Lecture 5, p.38. Writing ω in both systems as

$$\omega = \omega_i \, \lambda^i|_p = \tilde{\omega}_j \, \tilde{\lambda}^j|_p ,$$

we can use $(*_1)$ to compute ω_i in terms of $\tilde{\omega}_j$:

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \omega\left(\frac{\partial \tilde{x}^j}{\partial x^i}(p)\frac{\partial}{\partial \tilde{x}^j}\Big|_p\right)$$
$$= \frac{\partial \tilde{x}^j}{\partial x^i}(p)\,\omega\left(\frac{\partial}{\partial \tilde{x}^j}\Big|_p\right)$$
$$= \frac{\partial \tilde{x}^j}{\partial x^i}(p)\,\tilde{\omega}_j \qquad (*_2).$$

· <u>DEF. 8.2</u>: Let M be a smooth mnfd. <u>The cotangent bundle of M</u> is denoted by T^*M and is defined as the disjoint union

$$T^*M := \bigsqcup_{p\in M} T_p^*M.$$

There is a natural projection map

$$\pi : T^*M \longrightarrow M,\ \omega\in T_p^*M \longmapsto p.$$

As above, given any smooth local coordinates (x^i) on an open subset $U\subseteq M$, for each $p\in U$ we denote by $(\lambda^i|_p)$ the basis for T_p^*M dual to $(\partial/\partial x^i|_p)$. This defines n maps

$$\lambda^1,\dots,\lambda^n : U \longrightarrow T^*M$$

(to be denoted differently soon), called <u>coordinate covector fields</u>.

· <u>PROP. 8.3</u> (The cotangent bundle as a vector bundle): Let M be a smooth n-mnfd. With its standard projection map and the natural vector space structure on each fiber, the cotangent bundle T^*M has a unique topology and smooth structure making it into a smooth vector bundle of rank n over M for which all coordinate covector fields are smooth local sections.

PROOF: (Similar to the proof of PROP. 6.5) Given a smooth chart (U, φ) for M, with coordinate fncts (x^i), define

$$\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$
$$\xi_i \lambda^i|_p \longmapsto (p, (\xi_1, \dots, \xi_n)),$$

where λ^i is the i-th coordinate covector field associated with (x^i). Suppose that $(\tilde{U}, \tilde{\varphi})$ is another smooth chart for M with coordinate fncts $(\tilde{x}^j)$, and let $\tilde{\Phi} : \pi^{-1}(\tilde{U}) \longrightarrow \tilde{U} \times \mathbb{R}^n$ be defined analogously. On $\pi^{-1}(U \cap \tilde{U})$, it follows from $(*_2)$ that

$$(\Phi \circ \tilde{\Phi}^{-1})(p, (\tilde{\xi}^1, \dots, \tilde{\xi}^n)) = \left(p, \left(\frac{\partial \tilde{x}^j}{\partial x^1}(p) \tilde{\xi}_j, \dots, \frac{\partial \tilde{x}^j}{\partial x^n}(p) \tilde{\xi}_j\right)\right).$$

The $GL(n, \mathbb{R})$-valued fnct $(\partial \tilde{x}^j / \partial x^i)$ is smooth, so it follows from the vector bundle chart lemma (= LEM 6.7) that T^*M has a smooth structure making it into a smooth vector bundle for which the maps Φ are local trivializations. Uniqueness follows as in the proof of ES11E1. ∎

L12 ↑

↓ L13

As in the case of the tangent bundle, smooth local coordinates for M yield smooth local coordinates for its cotangent bundle. If (x^i) are smooth coordinates on an open subset $U \subseteq M$, ES10ES(d) shows that the map

$$\pi^{-1}(U) \longrightarrow \mathbb{R}^{2n}, \quad \xi_i \lambda^i|_p \longmapsto (x^1(p), \dots, x^n(p), \xi_1, \dots, \xi_n)$$

is a smooth coordinate chart for T^*M. We call (x^i, ξ_i) <u>the natural coordinates for T^*M associated with (x^i)</u>.