Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 12

Exercise 1:

(a) *Restricting smooth vector fields to submanifolds*: Let M be a smooth manifold, let S be an immersed submanifold of M, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. Prove the following assertions:

  (i) If $Y \in \mathfrak{X}(M)$ and if there is $X \in \mathfrak{X}(S)$ that is ι-related to Y, then $Y \in \mathfrak{X}(M)$ is tangent to S.

  (ii) If $Y \in \mathfrak{X}(M)$ is tangent to S, then there is a unique smooth vector field on S, denote by $Y|_S$, which is ι-related to Y.

  [Hint: Determine first the candidate vector field on S and then use *Theorem 5.6* and *Proposition 5.16* to show that it is smooth.]

(b) *Lie brackets of smooth vector fields tangent to submanifolds*: Let M be a smooth manifold and let S be an immersed submanifold of M. If Y_1 and Y_2 are smooth vector fields on M that are tangent to S, then show that their Lie bracket $[Y_1, Y_2]$ is also tangent to S.

Exercise 2:
Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon J \to M$ be an integral curve of V. Prove the following assertions:

(a) *Rescaling lemma*: For any $a \in \mathbb{R}$, the curve

$$\widetilde{\gamma}\colon \widetilde{J} \to M,\ t \mapsto \gamma(at)$$

is an integral curve of the vector field $\widetilde{V} := aV$ on M, where $\widetilde{J} := \{t \in \mathbb{R} \mid at \in J\}$.

(b) *Translation lemma*: For any $b \in \mathbb{R}$, the curve

$$\widehat{\gamma}\colon \widehat{J} \to M,\ t \mapsto \gamma(t+b)$$

is also an integral curve of V on M, where $\widehat{J} := \{t \in \mathbb{R} \mid t+b \in J\}$.

Exercise 3 (to be submitted by Friday, 15.12.2023, 20:00):

(a) Compute the Lie bracket $[X, Y]$ of each of the following pairs of smooth vector fields X and Y on $\mathbb{R}^3$:

(i) $X = y\,\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial y}$.

(ii) $X = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}$ and $Y = -z\,\frac{\partial}{\partial y} + y\,\frac{\partial}{\partial z}$.

(b) Compute the flow of each of the following smooth vector fields on $\mathbb{R}^2$:

(i) $V = x\,\frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial y}$.

(ii) $W = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

Exercise 4:
Let $\theta\colon \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. Show that the infinitesimal generator V of θ is a smooth vector field on M, and that each curve $\theta^{(p)}\colon \mathbb{R} \to M$ is an integral curve of V.

Exercise 5:

(a) *Naturality of flows*: Let $F\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η be the flow of Y. Show that if X and Y are F-related, then for each $t \in \mathbb{R}$ it holds that $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:

$$\begin{array}{ccc} M_t & \xrightarrow{F} & N_t \\ {\scriptstyle\theta_t}\downarrow & & \downarrow{\scriptstyle\eta_t} \\ M_{-t} & \xrightarrow{F} & N_{-t} \end{array}$$

(b) *Diffeomorphism invariance of flows*: Let $F\colon M \to N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$ and θ is the flow of X, then show that the flow of F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

Definition. Let V be a (rough) vector field on a smooth manifold M. A point $p \in M$ is called a *singular point* of V if $V_p = 0 \in T_pM$; otherwise, it is called a *regular point* of V.

Exercise 6:
Let V be a smooth vector field on a smooth manifold M and let $\theta\colon \mathfrak{D} \to M$ be the flow generated by V. Prove the following assertions:

(a) If $p \in M$ is a singular point of V, then $\mathfrak{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) \equiv p$.

(b) If $p \in M$ is a regular point of V, then $\theta^{(p)}\colon \mathfrak{D}^{(p)} \to M$ is a smooth immersion.

[Hint: Argue by contraposition and use the fundamental theorem on flows.]