Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 12 – Solutions

Exercise 1:

(a) *Restricting smooth vector fields to submanifolds*: Let M be a smooth manifold, let S be an immersed submanifold of M, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. Prove the following assertions:

(i) If $Y \in \mathfrak{X}(M)$ and if there is $X \in \mathfrak{X}(S)$ that is ι-related to Y, then $Y \in \mathfrak{X}(M)$ is tangent to S.

(ii) If $Y \in \mathfrak{X}(M)$ is tangent to S, then there is a unique smooth vector field on S, denote by $Y|_S$, which is ι-related to Y.

[Hint: Determine first the candidate vector field on S and then use *Theorem 5.6* and *Proposition 5.16* to show that it is smooth.]

(b) *Lie brackets of smooth vector fields tangent to submanifolds*: Let M be a smooth manifold and let S be an immersed submanifold of M. If Y_1 and Y_2 are smooth vector fields on M that are tangent to S, then show that their Lie bracket $[Y_1, Y_2]$ is also tangent to S.

Solution:

(a) Since X is ι-related to Y, it holds that $Y_p = d\iota_p(X_p)$ for all $p \in S$, which means that $Y_p \in T_pS$ for all $p \in S$, i.e., Y is tangent to S.

(b) Since by hypothesis we have $Y_p \in d\iota_p(T_pS)$ for all $p \in S$, we may define a rough vector field $X\colon S \to TS$ by requiring that, for any $p \in S$, $X_p \in T_pS$ is the unique vector such that $d\iota_p(X_p) = Y_p$. By the injectivity of $d\iota_p$, it is clear that X is unique, and that it is ι-related to Y, so it remains to show that X is smooth. To this end, let $p \in S$ be arbitrary. By *Proposition 5.16* there is an open neighborhood V of p in S such that V is embedded in M. By *Theorem 5.6* there exists a smooth chart $\big(U, (x^i)\big)$ for M such that $V \cap U$ is a k-slice in U – we may assume that $V \cap U$ is the slice given by $x^{k+1} = \ldots = x^n = 0$ – and $(x^1, \ldots, x^k)$ are local coordinates for S in $V \cap U$ Consider the coordinate representation

$$Y = \sum_i Y^i \frac{\partial}{\partial x^i}$$

of Y on U. By *Proposition 7.8* (evaluating the above expression at the coordinate function x^i with $i > k$) we infer that $Y^{k+1} = \ldots = Y^n = 0$ on $V \cap U$, since Y is tangent to S. Therefore,

$$X = \sum_{1 \le i \le k} Y^i|_{U\cap V} \frac{\partial}{\partial x^i}\bigg|_{U\cap V}$$

is the coordinate representation of X on $V \cap U$, and each $Y^i|_{U\cap V}$ is smooth by part (a) of *Exercise* 5, *Sheet* 8, so X is smooth on $U \cap V$, and we are done.

(Let us now verify for completeness that

$$X \stackrel{?}{=} \sum_{1 \le i \le k} Y^i|_{U\cap V} \frac{\partial}{\partial x^i}\bigg|_{U\cap V}$$

is the coordinate representation of X on $V \cap U$. Let $f \in C^\infty(U \cap V)$ be arbitrary, and consider the function

$$F := f \circ \psi^{-1} \circ \pi \circ \varphi \colon U \to \mathbb{R},$$

where $\varphi = (x^1, \ldots, x^n)$, $\psi = (x^1, \ldots, x^k)|_{U\cap V}$ and $\pi \colon \mathbb{R}^n \to \mathbb{R}^k$ is the projection onto the first k coordinates. Then F is smooth, as φ and ψ are smooth charts for M and V, respectively, and furthermore $F \circ \iota = f$, i.e., F is an extension of f to U. We have

$$X_p(f) = X_p(F \circ \iota) = d\iota_p(X_p)(F) = Y_p(F) = \sum_{1 \le i \le k} Y^i(p) \frac{\partial F}{\partial x^i}(p).$$

Now (you should convince yourself that) for all $1 \le i \le k$ we have

$$\frac{\partial F}{\partial x^i}(p) = \frac{\partial f}{\partial x^i}(p),$$

and thus

$$X_p(f) = \sum_{1 \le i \le k} Y^i|_{U\cap V}(p) \frac{\partial f}{\partial x^i}(p)$$

for any $p \in U \cap V$ and any f defined on a neighborhood of p in M.)

Exercise 2: Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma \colon J \to M$ be an integral curve of V. Prove the following assertions:

(a) *Rescaling lemma*: For any $a \in \mathbb{R}$, the curve

$$\widetilde{\gamma} \colon \widetilde{J} \to M,\ t \mapsto \gamma(at)$$

is an integral curve of the vector field $\widetilde{V} := aV$ on M, where $\widetilde{J} := \{t \in \mathbb{R} \mid at \in J\}$.

(b) *Translation lemma*: For any $b \in \mathbb{R}$, the curve

$$\widehat{\gamma} \colon \widehat{J} \to M,\ t \mapsto \gamma(t + b)$$

is also an integral curve of V on M, where $\widehat{J} := \{t \in \mathbb{R} \mid t + b \in J\}$.

Solution:

(a) If $t \in \widetilde{J}$, then

$$\widetilde{\gamma}'(t) = a\gamma'(at) = aV_{\gamma(at)} = \widetilde{V}_{\widetilde{\gamma}(t)}.$$

(b) If $t \in \widehat{J}$, then

$$\widehat{\gamma}'(t) = \gamma'(t+b) = V_{\gamma(t+b)} = V_{\widehat{\gamma}(t)}.$$

Exercise 3:

(a) Compute the Lie bracket $[X, Y]$ of each of the following pairs of smooth vector fields X and Y on $\mathbb{R}^3$:

(i) $X = y\,\dfrac{\partial}{\partial z} - 2xy^2\,\dfrac{\partial}{\partial y}$ and $Y = \dfrac{\partial}{\partial y}$.

(ii) $X = -y\,\dfrac{\partial}{\partial x} + x\,\dfrac{\partial}{\partial y}$ and $Y = -z\,\dfrac{\partial}{\partial y} + y\,\dfrac{\partial}{\partial z}$.

(b) Compute the flow of each of the following smooth vector fields on $\mathbb{R}^2$:

(i) $V = x\,\dfrac{\partial}{\partial x} + 2y\,\dfrac{\partial}{\partial y}$.

(ii) $W = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

Solution:

(a) In case (i), writing

$$X = 0\,\frac{\partial}{\partial x} - 2xy^2\,\frac{\partial}{\partial y} + y\,\frac{\partial}{\partial z}$$

and

$$Y = 0\,\frac{\partial}{\partial x} + 1\,\frac{\partial}{\partial x} + 0\,\frac{\partial}{\partial z},$$

by invoking part (a) of *Exercise* 5, *Sheet* 11 we compute that

$$[X, Y] = 4xy\,\frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

In case (ii), we similarly obtain

$$[X, Y] = -z\,\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}.$$

(b) We first deal with (i); we argue exactly as in the solution to part (c) of *Exercise* 3, *Sheet* 11. Observe first that the unique maximal integral curve of V starting at $p = (0,0)$ is the constant curve $\gamma_0\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (0,0)$; see *Exercise* 6(a). Now, if $\gamma\colon J \to \mathbb{R}^2$ is a smooth curve, written in standard coordinates as $\gamma(t) = \big(\gamma^1(t), \gamma^2(t)\big)$, then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V translates to

$$\dot{\gamma}^1(t) = \gamma^1(t),$$
$$\dot{\gamma}^2(t) = 2\gamma^2(t).$$

Therefore, there are constants $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{aligned}\gamma^1 &\colon J = \mathbb{R} \to \mathbb{R},\ \gamma^1(t) = c_1 e^t,\\ \gamma^2 &\colon J = \mathbb{R} \to \mathbb{R},\ \gamma^2(t) = c_1 e^{2t},\end{aligned}$$

so the unique maximal integral curve of V starting at $p = (p^1, p^2) \in \mathbb{R}^2$ is the smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \big(p^1 e^t, p^2 e^{2t}\big)$, which in passing is a smooth immersion for $p \in \mathbb{R}^2 \setminus \{(0,0)\}$; see *Exercise* 6(b). In conclusion, V is a complete vector field on $\mathbb{R}^2$ whose flow is the map

$$\theta_V \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,\ \big(t,(x,y)\big) \mapsto \big(xe^t, ye^{2t}\big).$$

We now deal with (ii). Working as in (i), we find that the unique maximal integral curve of W starting at $p = (p^1, p^2) \in \mathbb{R}^2$ is the smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \big(p^1 e^t, p^2 e^{-t}\big)$, which is a smooth immersion for $p \in \mathbb{R}^2 \setminus \{(0,0)\}$. Hence, the flow of the complete vector field W is the map

$$\theta_W \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,\ \big(t,(x,y)\big) \mapsto \big(xe^t, ye^{-t}\big).$$

Exercise 4: Let $\theta\colon \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. Show that the infinitesimal generator V of θ is a smooth vector field on M, and that each curve $\theta^{(p)}\colon \mathbb{R} \to M$ is an integral curve of V.

Solution: By definition of the infinitesimal generator, we have

$$V_p = \left.\frac{d}{dt}\right|_{t=0} \theta(t,p) \quad \text{for all } p \in M. \tag{$\star$}$$

First, to show that V is smooth, we apply *Proposition 7.5*(c): Given an open subset U of M, a smooth real-valued function f on U, and $p \in U$, we have

$$\begin{aligned} Vf(p) = V_p f &= \left(\left.\frac{d}{dt}\right|_{t=0} \theta(t,p)\right) f \\ &= \left.\frac{d}{dt}\right|_{t=0} (f\circ\theta)(t,p) = \left.\frac{\partial}{\partial t}\right|_{(0,p)} (f\circ\theta)(t,p). \end{aligned}$$

Since the composite map $f \circ \theta$ is smooth, its partial derivative with respect to t is smooth as well. Thus, $Vf(p)$ depends smoothly on p, which implies that V is smooth.

Next, fix $p \in M$ and $s \in \mathbb{R}$. We have to show that

$$\left.\frac{d}{dt}\right|_{t=s} \theta(t,p) = V_{\theta(s,p)} \overset{(\star)}{=} \left.\frac{d}{dt}\right|_{t=0} \theta\big(t, \theta(s,p)\big).$$

By definition of a flow, we have

$$\theta(t+s,p) = \theta\big(t,\theta(s,p)\big),$$

and by first differentiating the above relation with respect to t and then evaluating at $t = 0$ we obtain the required identity.

Exercise 5:

(a) *Naturality of flows*: Let $F\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η be the flow of Y. Show that if X and Y are F-related, then for each $t \in \mathbb{R}$ it holds that $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:

$$\begin{array}{ccc} M_t & \xrightarrow{\;F\;} & N_t \\ {\scriptstyle \theta_t}\big\downarrow & & \big\downarrow{\scriptstyle \eta_t} \\ M_{-t} & \xrightarrow{\;F\;} & N_{-t} \end{array}$$

[Hint: Use part (e) of *Exercise* 4, *Sheet* 11.]

(b) *Diffeomorphism invariance of flows*: Let $F\colon M \to N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$ and θ is the flow of X, then show that the flow of F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

Solution:

(a) Denote by $\mathcal{D}_X$ resp. $\mathcal{D}_Y$ the flow domain of θ resp. η. Fix $t \in \mathbb{R}$ and let $p \in M_t$. Then $t \in \mathcal{D}_X^{(p)}$ and $\theta^{(p)}\colon \mathcal{D}_X^{(p)} \to M$ is the unique maximal integral curve of X starting at p. By part (e) of *Exercise* 4, *Sheet* 11, $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F(p)$. Hence, by maximality, we obtain that $\mathcal{D}_X^{(p)} \subseteq \mathcal{D}_Y^{(F(p))}$, and thus $t \in \mathcal{D}_Y^{(F(p))}$, which shows that $F(p) \in N_t$. In conclusion, $F(M_t) \subseteq N_t$. Finally, we have

$$F \circ \theta_t(p) = F\big(\theta(t,p)\big) \overset{(*)}{=} \eta\big(t, F(p)\big) = \eta_t \circ F(p),$$

where in $(*)$ we again used that $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F(p)$ and thus it is equal to $\eta^{(F(p))}$ where its defined (this uses the uniqueness part in the theorem about solutions to a system of ODEs).

(b) Denote by η the flow of F_*X. Applying part (a) to both F and F^{-1} we obtain that $F(M_t) \subseteq N_t$ and $F^{-1}(N_t) \subseteq M_t$, so that $F(M_t) = N_t$. Furthermore, the commutativity of the above diagram shows that $\eta_t = F \circ \theta_t \circ F^{-1}$ for all $t \in \mathbb{R}$.

Exercise 6: Let V be a smooth vector field on a smooth manifold M and let $\theta\colon \mathfrak{D} \to M$ be the flow generated by V. Prove the following assertions:

(a) If $p \in M$ is a singular point of V, then $\mathfrak{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) \equiv p$.

(b) If $p \in M$ is a regular point of V, then $\theta^{(p)}\colon \mathfrak{D}^{(p)} \to M$ is a smooth immersion.

[Hint: Argue by contraposition and use the fundamental theorem on flows.]

Solution:

(a) If $V_p = 0$, then the constant curve $\gamma\colon \mathbb{R} \to M,\ t \mapsto p$ is clearly an integral curve of V, so it must be equal to $\theta^{(p)}$ by uniqueness and maximality.

(b) Assume that $\theta^{(p)} \colon \mathfrak{D}^{(p)} \to M$ is not a smooth immersion. Then $\theta^{(p)\prime}(s) = 0$ for some $s \in \mathfrak{D}^{(p)}$. Set $q := \theta^{(p)}(s)$ and note that $V_q = 0$, since $\theta^{(p)}$ is an integral curve of V. Thus, q is a singular point of V, and by part (a) we infer that $\mathfrak{D}^{(q)} = \mathbb{R}$ and that $\theta^{(q)}$ is the constant curve $\theta^{(q)}(t) \equiv q$. It follows from *Theorem 7.14* (b) that $\mathfrak{D}^{(p)} = \mathbb{R}$ as well, and for all $t \in \mathbb{R}$ the group law gives

$$\theta^{(p)}(t) = \theta_t(p) = \theta_{t-s}\big(\theta(s,p)\big) = \theta_{t-s}(q) = q.$$

For $t = 0$ we obtain $q = \theta^{(p)}(0) = p$, and hence $\theta^{(p)}(t) \equiv p$ and $V_p = \theta^{(p)\prime}(0) = 0$, which contradicts the assumption that p is a regular point of V. This finishes the proof of (b).