Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
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Exercise Sheet 13 – Part II

Exercise 1:
Let $F\colon M \to N$ be a smooth map. Prove the following assertions:

(a) $F^*\colon \Omega^k(N) \to \Omega^k(M)$ is an $\mathbb{R}$-linear map.

(b) It holds that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart $\big(V, (y^i)\big)$ for N, we have

$$F^*\left(\sum_I{}' \omega_I dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\, d\left(y^{i_1} \circ F\right) \wedge \ldots \wedge d\left(y^{i_k} \circ F\right).$$

Exercise 2:
Let (r, θ) be polar coordinates on the right half-plane $H = \big\{(x, y) \mid x > 0\big\}$. Compute the polar coordinate expression for the smooth 1-form $x\,dy - y\,dx \in \Omega^1(\mathbb{R}^2)$ and for the smooth 2-form $dx \wedge dy \in \Omega^2(\mathbb{R}^2)$.

[Hint: Think of the change of coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ as the coordinate expression for the identity map of H, but using (r, θ) as coordinates for the domain and (x, y) as coordinates for the codomain.]

Exercise 3 (to be submitted by Friday, 22.12.2023, 20:00):

(a) Let M be a compact, connected, smooth manifold of dimension $n > 0$. Show that every exact smooth covector field on M vanishes at least at two points of M.

(b) Let V be a finite-dimensional real vector space and let $\omega^1, \ldots, \omega^k \in V^*$. Show that the covectors $\omega^1, \ldots, \omega^k$ are linearly dependent if and only if $\omega^1 \wedge \ldots \wedge \omega^k = 0$.

(c) Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (s, t) \mapsto (st, e^t)$$

and the smooth covector field $\omega \in \mathfrak{X}^*(\mathbb{R}^2)$ given by

$$\omega = xdy.$$

Compute $d\omega$ and $F^*\omega$, and verify by direct computation that $d(F^*\omega) = F^*(d\omega)$.

Exercise 4:
Consider the smooth 2-form

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

on $\mathbb{R}^3$ with standard coordinates (x, y, z).

(a) Compute ω in spherical coordinates for $\mathbb{R}^3$ defined by

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

(b) Compute $d\omega$ in spherical coordinates.

(c) Consider the inclusion map $\iota\colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and compute the pullback $\iota^*\omega$ to $\mathbb{S}^2$, using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota^*\omega$ is nowhere zero.

Definition. Let M be a smooth n-manifold. Given a local frame $(E_1, \ldots, E_n)$ for M over an open subset $U \subseteq M$, there is a uniquely determined (rough) local coframe $(\varepsilon^1, \ldots, \varepsilon^n)$ over U such that $(\varepsilon^i|_p)$ is the dual basis to $(E_i|_p)$ for each $p \in U$, or equivalently $\varepsilon^i(E_j) = \delta^i_j$. This coframe is called *the coframe dual to* (E_i). Conversely, if we start with a local coframe (ε^i) for M over an open subset $U \subseteq M$, there is a uniquely determined (rough) local frame (E_i), called *the frame dual to* (ε^i), determined by $\varepsilon^i(E_j) = \delta^i_j$. For example, in a smooth chart, the coordinate frame $(\partial/\partial x^i)$ and the coordinate coframe (dx^i) are dual to each other. It can easily be shown that (E_i) is smooth if and only if (ε^i) is smooth.

Exercise 5:

(a) *Exterior derivative of a smooth* 1*-form*: Show that for any smooth 1-form ω and any smooth vector fields X and Y on a smooth manifold M it holds that

$$d\omega(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X, Y]\big).$$

(b) Let M be a smooth n-manifold, let (E_i) be a smooth local frame for M and let (ε^i) be the dual coframe. For each i, denote by b^i_{jk} the component functions of the exterior derivative of ε^i in this frame, and for each j, k, denote by c^i_{jk} the component functions of the Lie bracket $[E_j, E_k]$:

$$d\varepsilon^i = \sum_{j<k} b^i_{jk}\, \varepsilon^j \wedge \varepsilon^k \quad \text{and} \quad [E_j, E_k] = c^i_{jk}\, E_i.$$

Show that $b^i_{jk} = -c^i_{jk}$.

Exercise 6:

(a) Let M be a smooth manifold and let $\omega \in \Omega^1(M) = \mathfrak{X}^*(M)$. Show that the following are equivalent:

(i) ω is closed.

(ii) ω satisfies

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j}$$

in some smooth chart $\big(U, (x^i)\big)$ around every point $p \in M$.

(iii) For any open subset $U \subseteq M$ and any $X, Y \in \mathfrak{X}(U)$, we have

$$X\big(\omega(Y)\big) - Y\big(\omega(X)\big) = \omega\big([X, Y]\big).$$

(b) Consider the smooth covector fields

$$\omega = y \cos(xy)\, dx + x \cos(xy)\, dy \in \mathfrak{X}^*(\mathbb{R}^2)$$

and

$$\eta = x \cos(xy)\, dx + y \cos(xy)\, dy \in \mathfrak{X}^*(\mathbb{R}^2).$$

Show that ω is closed and exact, whereas η is neither closed nor exact.