Differential Geometry II - Smooth Manifolds
Winter Term 2023/2024
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 13 – Part II – Solutions

Exercise 1: Let $F\colon M \to N$ be a smooth map. Prove the following assertions:

(a) $F^*\colon \Omega^k(N) \to \Omega^k(M)$ is an $\mathbb{R}$-linear map.

(b) It holds that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart $\big(V, (y^i)\big)$ on N, we have

$$F^*\left(\sum_I{}' \omega_I dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\, d\left(y^{i_1} \circ F\right) \wedge \ldots \wedge d\left(y^{i_k} \circ F\right).$$

Solution:

(a) Let $\omega, \eta \in \Omega^k(N)$ and $\lambda, \mu \in \mathbb{R}$. Fix $p \in M$ and let $v_1, \ldots, v_k \in T_pM$. We have

$$\begin{aligned}
\big(F^*(\lambda\omega + \mu\eta)\big)_p(v_1, \ldots, v_k) &= (\lambda\omega + \mu\eta)_p\big(dF_p(v_1), \ldots, dF_p(v_k)\big) \\
&= \lambda\, \omega_p\big(dF_p(v_1), \ldots, dF_p(v_k)\big) + \mu\, \eta_p\big(dF_p(v_1), \ldots, dF_p(v_k)\big) \\
&= \lambda(F^*\omega)_p(v_1, \ldots, v_k) + \mu(F^*\eta)_p(v_1, \ldots, v_k) \\
&= \big(\lambda(F^*\omega)_p + \mu(F^*\eta)_p\big)(v_1, \ldots, v_k),
\end{aligned}$$

which implies that

$$\big(F^*(\lambda\omega + \mu\eta)\big)_p = \lambda(F^*\omega)_p + \mu(F^*\eta)_p,$$

and whence $F^*\colon \Omega^k(N) \to \Omega^k(M)$ is an $\mathbb{R}$-linear map.

(b) Assume that ω resp. η are k- resp. l-covectors. Fix $p \in M$ and let $v_1, \ldots, v_{k+l} \in T_pM$. We have

$$\begin{aligned}
&F^*(\omega \wedge \eta)_p\,(v_1, \ldots, v_{k+l}) = (\omega \wedge \eta)_{F(p)}\big(dF_p(v_1), \ldots, dF_p(v_{k+l})\big) \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\, \omega\left(dF_p\big(v_{\sigma(1)}\big), \ldots, dF_p\big(v_{\sigma(k)}\big)\right) \eta\left(dF_p\big(v_{\sigma(k+1)}\big), \ldots, dF_p\big(v_{\sigma(k+l)}\big)\right)
\end{aligned}$$

and

$$
\begin{aligned}
&[(F^*\omega) \wedge (F^*\eta)]_p \, (v_1, \ldots, v_{k+l}) = \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \, (F^*\omega)_p \left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) (F^*\eta)_p \left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\right) \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \, \omega \left(dF_p\big(v_{\sigma(1)}\big), \ldots, dF_p\big(v_{\sigma(k)}\big)\right) \eta \left(dF_p\big(v_{\sigma(k+1)}\big), \ldots, dF_p\big(v_{\sigma(k+l)}\big)\right).
\end{aligned}
$$

As the two expressions agree, we conclude that

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta).$$

(c) The assertion follows immediately from (a), (b) and *Proposition 8.11.*

Exercise 2: Let (r, θ) be polar coordinates on the right half-plane $H = \big\{(x, y) \mid x > 0\big\}$. Compute the polar coordinate expression for the smooth 1-form $x\,dy - y\,dx \in \Omega^1(\mathbb{R}^2)$ and for the smooth 2-form $dx \wedge dy \in \Omega^2(\mathbb{R}^2)$.

[Hint: Think of the change of coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ as the coordinate expression for the identity map of H, but using (r, θ) as coordinates for the domain and (x, y) as coordinates for the codomain.]

Solution: We have

$$
\begin{aligned}
\mathrm{Id}^*(x\,dy - y\,dx) &= r\cos\theta\, d(r\sin\theta) - r\sin\theta\, d(r\cos\theta) \\
&= r\cos\theta(\sin\theta\, dr + r\cos\theta\, d\theta) - r\sin\theta(\cos\theta\, dr - r\sin\theta\, d\theta) \\
&= r^2\cos^2\theta\, d\theta + r^2\sin^2\theta\, d\theta \\
&= r^2\, d\theta
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Id}^*(dx \wedge dy) &= d(r\cos\theta) \wedge d(r\sin\theta) \\
&= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) \\
&= r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr \\
&= r\, dr \wedge d\theta,
\end{aligned}
$$

since $dr \wedge dr = 0 = d\theta \wedge d\theta$ and $dr \wedge d\theta = -d\theta \wedge dr$.

Exercise 3:

(a) Let M be a compact, connected, smooth manifold of dimension $n > 0$. Show that every exact smooth covector field on M vanishes at least at two points of M.

(b) Let V be a finite-dimensional real vector space and let $\omega^1, \ldots, \omega^k \in V^*$. Show that the covectors $\omega^1, \ldots, \omega^k$ are linearly dependent if and only if $\omega^1 \wedge \ldots \wedge \omega^k = 0$.

(c) Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (s,t) \mapsto (st, e^t)$$

and the smooth covector field $\omega \in \mathfrak{X}^*(\mathbb{R}^2)$ given by

$$\omega = xdy.$$

Compute $d\omega$ and $F^*\omega$, and verify by direct computation that $d(F^*\omega) = F^*(d\omega)$.

Solution:

(a) Let $\omega \in \mathfrak{X}^*(M)$ be exact and let $f \in C^\infty(M)$ such that $\omega = df$. Since M is compact, f attains its minimum at a point $p \in M$ and its maximum at a point $q \in M$, and since df is represented in coordinates by the gradient of (the coordinate representation of) f, we have $df_p = 0 = df_q$. Note also that if $p = q$, then f is constant, and thus $df = 0$ by *Exercise 2*(e) from *Exercise Sheet 13 – Part I.*

(b) Assume first that the covectors $\omega^1, \dots, \omega^k$ are linearly dependent. Then there exist $j \in \{1, \dots, k\}$ and $\lambda_1, \dots, \widehat{\lambda_j}, \dots, \lambda_k \in \mathbb{R}$ such that $\omega^j = \sum_{i \neq j} \lambda_i \omega^i$. Therefore,

$$\begin{aligned}
\omega^1 \wedge \ldots \wedge \omega^{j-1} \wedge \omega^j \wedge \omega^{j+1} \wedge \ldots \wedge \omega^k &= \omega^1 \wedge \ldots \wedge \omega^{j-1} \wedge \sum_{i \neq j} \lambda_i \omega^i \wedge \omega^{j+1} \wedge \ldots \wedge \omega^k \\
&= \sum_{i \neq j} \omega^1 \wedge \ldots \wedge \omega^{j-1} \wedge \omega^i \wedge \omega^{j+1} \wedge \ldots \wedge \omega^k \\
&= 0
\end{aligned}$$

by [Multilinear Algebra, Proposition 24(d)].

Assume now that the covectors $\omega^1, \dots, \omega^k$ are linearly independent. We will show below that (the alternating k-multilinear function) $\eta := \omega^1 \wedge \ldots \wedge \omega^k \neq 0$. It suffices to find $v_1, \dots, v_k \in V$ such that $\eta(v_1, \dots, v_k) \neq 0$. To this end, set $n = \dim_{\mathbb{R}} V$ and note that $n \geq k$. Since $\omega^1, \dots, \omega^k$ are linearly independent elements of V^*, we can complete them to a basis $\{\omega^1, \dots, \omega^k, \omega^{k+1}, \dots, \omega^n\}$ of V^*, and consider subsequently the basis $\{v_1, \dots, v_n\}$ of V dual to $\{\omega^j\}$; see (the second paragraph after) [Multilinear Algebra, Proposition 4]. By [Multilinear Algebra, Proposition 24(d)] we then obtain

$$\eta(v_1, \dots, v_k) = \det\Big(\big(\omega^j(v_i)\big)\Big) = \det\big(\delta_i^j\big) = 1,$$

and thus $\eta \neq 0$, as desired.

(c) We have

$$d\omega = dx \wedge dy$$

and

$$F^*\omega = (st)\, d(e^t) = ste^t dt.$$

Therefore,

$$d(F^*\omega) = d(ste^t) \wedge dt = (te^t\, ds + s(1+t)e^t\, dt) \wedge dt = te^t\, ds \wedge dt$$

and

$$F^*(d\omega) = d(st) \wedge d(e^t) = (t\, ds + s\, dt) \wedge (e^t\, dt) = te^t\, ds \wedge dt.$$

Exercise 4: Consider the smooth 2-form

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

on $\mathbb{R}^3$ with standard coordinates (x, y, z).

(a) Compute ω in spherical coordinates for $\mathbb{R}^3$ defined by

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

(b) Compute $d\omega$ in spherical coordinates.

(c) Consider the inclusion map $\iota\colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and compute the pullback $\iota^*\omega$ to $\mathbb{S}^2$, using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota^*\omega$ is nowhere zero.

Solution:

(a) We have

$$\begin{aligned} dx &= d(\rho \sin\varphi \cos\theta) = \sin\varphi \cos\theta\ d\rho + \rho \cos\varphi \cos\theta\ d\varphi - \rho \sin\varphi \sin\theta\ d\theta, \\ dy &= d(\rho \sin\varphi \sin\theta) = \sin\varphi \sin\theta\ d\rho + \rho \cos\varphi \sin\theta\ d\varphi + \rho \sin\varphi \cos\theta\ d\theta, \\ dz &= d(\rho \cos\varphi) = \cos\varphi\ d\rho - \rho \sin\varphi\ d\varphi. \end{aligned}$$

Therefore, one computes that

$$\begin{aligned} dy \wedge dz &= \rho^2 \sin^2\varphi \cos\theta\ d\varphi \wedge d\theta + \rho \sin\varphi \cos\varphi \cos\theta\ d\theta \wedge d\rho - \rho \sin\theta\ d\rho \wedge d\varphi, \\ dz \wedge dx &= \rho^2 \sin^2\varphi \sin\theta\ d\varphi \wedge d\theta + \rho \sin\varphi \cos\varphi \sin\theta\ d\theta \wedge d\rho + \rho \cos\theta\ d\rho \wedge d\varphi, \\ dx \wedge dy &= \rho^2 \cos\varphi \sin\varphi\ d\varphi \wedge d\theta - \rho \sin^2\varphi\ d\theta \wedge d\rho. \end{aligned}$$

By combining these expressions, we thus obtain

$$\begin{aligned} \omega &= x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \\ &= (\rho^3 \sin^3\varphi \cos^2\theta + \rho^3 \sin^3\varphi \sin^2\theta + \rho^3 \cos^2\varphi \sin\varphi)\ d\varphi \wedge d\theta \\ &\quad + \underbrace{(\rho^2 \sin^2\varphi \cos\varphi \cos^2\theta + \rho^2 \sin^2\varphi \cos\varphi \sin^2\theta - \rho^2 \sin^2\varphi \cos\varphi)}_{=0}\ d\theta \wedge d\rho \\ &\quad + \underbrace{(-\rho^2 \sin\varphi \sin\theta \cos\theta + \rho^2 \sin\varphi \sin\theta \cos\theta)}_{=0}\ d\rho \wedge d\varphi \\ &= \rho^3 \sin\varphi \underbrace{(\sin^2\varphi \cos^2\theta + \sin^2\varphi \sin^2\theta + \cos^2\varphi)}_{=1}\ d\varphi \wedge d\theta \\ &= \rho^3 \sin\varphi\ d\varphi \wedge d\theta. \end{aligned}$$

(b) We have

$$d\big(\rho^3 \sin\varphi\big) = 3\rho^2 \sin\varphi\ d\rho + \rho^3 \cos\varphi\ d\varphi,$$

so we obtain

$$d\omega = d\big(\rho^3 \sin\varphi\big) \wedge d\varphi \wedge d\theta = 3\rho^2 \sin\varphi\ d\rho \wedge d\varphi \wedge d\theta.$$

Another way to compute $d\omega$ would be to note that

$$d\omega = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy = 3dx \wedge dy \wedge dz.$$

For the standard top differential form $dx \wedge dy \wedge dz$ on $\mathbb{R}^3$, a change of coordinates induces a factor given by the determinant of the Jacobian. You may remember or look up (or compute) that the determinant of the Jacobian of spherical coordinates is $\rho^2 \sin\varphi$, so we obtain $d\omega = 3\rho^2 \sin\varphi \; d\rho \wedge d\varphi \wedge d\theta$ as well.

(c) We just have to put $\rho = 1$ in the result of part (a). To justify precisely what is going on, let us spell this out in detail. Note that the change into spherical coordinates is provided by the diffeomorphism

$$\begin{aligned} G \colon \mathbb{R}_{>0} \times (0,\pi) \times (0, 2\pi) &\to U \subseteq \mathbb{R}^3 \\ (\rho, \varphi, \theta) &\mapsto (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi), \end{aligned}$$

where $V = \{(x,y,z) \mid y \neq 0\}$. So what we computed above is $G^*(\omega|_V)$. Note that spherical coordinates on the sphere are provided by the diffeomorphism

$$\begin{aligned} F \colon (0,\pi) \times (0, 2\pi) &\to V \subseteq \mathbb{S}^2 \\ (\varphi, \theta) &\mapsto (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi), \end{aligned}$$

where $U = \mathbb{S}^2 \cap V$. If we denote by j the embedding

$$\begin{aligned} j \colon (0,\pi) \times (0, 2\pi) &\to \mathbb{R}_{>0} \times (0,\pi) \times (0, 2\pi) \\ (\varphi, \theta) &\mapsto (1, \varphi, \theta), \end{aligned}$$

then this is precisely set up so that $G \circ j = \iota \circ F$. What we want to compute is $F^*\iota^*(\omega|_V)$, and this is given by

$$\begin{aligned} F^*\iota^*(\omega|_V) &= (\iota \circ F)^*(\omega|_V) = (G \circ j)^*(\omega|_V) = j^* G^*(\omega|_V) \\ &= j^*(\rho^3 \sin\varphi \; d\varphi \wedge d\theta) \\ &= \sin\varphi \; d\varphi \wedge d\theta. \end{aligned}$$

(d) As $\sin\varphi \neq 0$ for $\varphi \in (0,\pi)$, we infer that $F^*\iota^*(\omega|_V) = \sin\varphi \; d\varphi \wedge d\theta$ is nowhere vanishing on $(0,\pi) \times (0,2\pi)$. As F is an isomorphism, we obtain that $\iota^*(\omega|_V) = (\iota^*\omega)|_U$ is nowhere vanishing on U, i.e., at the points of $\mathbb{S}^2$ where $y \neq 0$. To conclude, note that we can do the exact same calculations for spherical coordinates around the x- and y-axes, and obtain that then $\iota^*\omega$ is non-zero also at all points where $z \neq 0$ resp. $x \neq 0$. Hence, $\iota^*\omega$ is nowhere zero.

Exercise 5:

(a) *Exterior derivative of a smooth 1-form*: Show that for any smooth 1-form ω and any smooth vector fields X and Y on a smooth manifold M it holds that

$$d\omega(X,Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X,Y]\big).$$

(b) Let M be a smooth n-manifold, let (E_i) be a smooth local frame for M and let (ε^i) be the dual coframe. For each i, denote by b^i_{jk} the component functions of the exterior derivative of ε^i in this frame, and for each j, k, denote by c^i_{jk} the component functions of the Lie bracket $[E_j, E_k]$:

$$d\varepsilon^i = \sum_{j<k} b^i_{jk}\, \varepsilon^j \wedge \varepsilon^k \quad \text{and} \quad [E_j, E_k] = c^i_{jk}\, E_i.$$

Show that $b^i_{jk} = -c^i_{jk}$.

Solution:

(a) Choose local coordinates $\big(U, (x^i)\big)$ and write

$$\omega = \sum_i c_i\, dx^i, \qquad X = \sum_i f_i \frac{\partial}{\partial x^i}, \qquad Y = \sum_i g_i \frac{\partial}{\partial x^i}.$$

Let $p \in M$ be arbitrary. Then

$$\begin{aligned}[d\omega(X,Y)](p) = (d\omega)_p(X_p, Y_p) &= \sum_i [(dc_i)_p \wedge (dx^i)_p](X_p, Y_p) \\ &= \sum_i \left[(dc_i)_p(X_p)(dx^i)_p(Y_p) - (dc_i)_p(Y_p)(dx^i)_p(X_p)\right] \\ &= \sum_{i,j} \left[g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) - f_i(p) g_j(p) \frac{\partial c_i}{\partial x^j}(p)\right].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left[X\big(\omega(Y)\big)\right](p) = \sum_i [X(c_i g_i)](p) &= \sum_i g_i(p)[X(c_i)](p) + c_i(p)[X(g_i)](p) \\ &= \sum_{i,j} \left[g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) + c_i(p) f_j(p) \frac{\partial g_i}{\partial x^j}(p)\right]\end{aligned}$$

and

$$\begin{aligned}\left[Y\big(\omega(X)\big)\right](p) = \sum_i [Y(c_i f_i)](p) &= \sum_i f_i(p)[Y(c_i)](p) + c_i(p)[Y(f_i)](p) \\ &= \sum_{i,j} \left[f_i(p) g_j(p) \frac{\partial c_i}{\partial x^j}(p) + c_i(p) g_j(p) \frac{\partial f_i}{\partial x^j}(p)\right]\end{aligned}$$

as well as

$$\left[\omega\big([X,Y]\big)\right](p) = \left[\sum_{i,j} c_i(p) f_j(p) \frac{\partial g_i}{\partial x^j}(p) - c_i(p) g_j(p) \frac{\partial f_i}{\partial x^j}(p)\right]$$

where we used part (a) of *Exercise* 5, *Sheet* 11. By combining these expressions, we obtain

$$\begin{aligned}
\big[X\big(\omega(Y)\big) - Y\big(\omega(X)\big)\big]\,(p) &= \left(\sum_{i,j} g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) + c_i(p) f_j(p) \frac{\partial g_i}{\partial x^j}(p)\right) \\
&\quad - \left(\sum_{i,j} f_i(p) g_j(p) \frac{\partial c_i}{\partial x^j}(p) + c_i(p) g_j(p) \frac{\partial f_i}{\partial x^j}(p)\right) \\
&= \left(\sum_{i,j} g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) - f_i(p) g_j(p) \frac{\partial c_i}{\partial x^j}(p)\right) \\
&\quad + \left(\sum_{i,j} c_i(p) f_j(p) \frac{\partial g_i}{\partial x^j}(p) - c_i(p) g_j(p) \frac{\partial f_i}{\partial x^j}(p)\right) \\
&= [d\omega(X,Y)](p) + \big[\omega\big([X,Y]\big)\big]\,(p).
\end{aligned}$$

This yields the assertion.

(b) Let us compute $d\varepsilon^i(E_j, E_k)$ for some i, j, k with $j < k$. By part (a) we obtain

$$\begin{aligned}
d\varepsilon^i(E_j, E_k) &= E_j\big(\varepsilon^i(E_k)\big) - E_k\big(\varepsilon^i(E_j)\big) - \varepsilon^i\big([E_j, E_k]\big) \\
&= \underbrace{E_j(\delta_{ik})}_{=0} - \underbrace{E_k(\delta_{ij})}_{=0} - c^i_{jk} = -c^i_{jk},
\end{aligned}$$

where in the last step we used that a derivation evaluated at a constant function gives 0. On the other hand, we have

$$d\varepsilon^i(E_j, E_k) = \sum_{j'<k'} b^i_{j'k'} \left[\varepsilon^{j'} \wedge \varepsilon^{k'}\right](E_j, E_k) = b^i_{jk},$$

where we used that $\varepsilon^{j'} \wedge \varepsilon^{k'} = \varepsilon^{(j',k')}$; see [Multilinear Algebra, Lemma 20(c) and Proposition 25(c)]. Hence, $b^i_{jk} = -c^i_{jk}$.

Remark.

(1) *Exercise* 5(b) shows that the exterior derivative is in a certain sense dual to the Lie bracket. In particular, it shows that if we know all the Lie brackets of basis vector fields in a smooth local frame, we can compute the exterior derivatives of the dual covector fields, and vice versa.

(2) There is an analogue of *Exercise* 5(a) for smooth k-forms as well, which is referred to as the *invariant formula for the exterior derivative* in the literature. Specifically, if $\omega \in \Omega^k(M)$, then for any $X_1, \dots, X_k \in \mathfrak{X}(M)$ it holds that

$$\begin{aligned}
d\omega(X_1, \dots, X_{k+1}) = &\sum_{1 \le i \le k+1} (-1)^{i-1} X_i\big(\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\big) + \\
&+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}\big),
\end{aligned}$$

where the hats indicate omitted arguments. It is worthwhile to mention that the above formula can be used to give an invariant definition of d, as well as an alternative proof of *Theorem 8.21* on the existence, uniqueness, and properties of d.

Exercise 6:

(a) Let M be a smooth manifold and let $\omega \in \Omega^1(M) = \mathfrak{X}^*(M)$. Show that the following are equivalent:

(i) ω is closed.

(ii) ω satisfies
$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j}$$
in some smooth chart $\big(U, (x^i)\big)$ around every point $p \in M$.

(iii) For any open subset $U \subseteq M$ and any $X, Y \in \mathfrak{X}(U)$, we have
$$X\big(\omega(Y)\big) - Y\big(\omega(X)\big) = \omega\big([X,Y]\big).$$

(b) Consider the smooth covector fields
$$\omega = y\cos(xy)\,dx + x\cos(xy)\,dy \in \mathfrak{X}^*(\mathbb{R}^2)$$
and
$$\eta = x\cos(xy)\,dx + y\cos(xy)\,dy \in \mathfrak{X}^*(\mathbb{R}^2).$$
Show that ω is closed and exact, whereas η is neither closed nor exact.

Solution:

(a) We will show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i)$\Rightarrow$(iii): Let us suppose that ω is closed, i.e. $d\omega = 0$. Let X, Y be smooth vector fields over some open subset $U \subseteq M$. By part (a) of *Exercise* 5, we have
$$0 = d\omega = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X,Y]\big),$$
so (iii) follows.

(iii) $\Rightarrow$ (ii) Suppose that (iii) holds. Let $p \in M$ be arbitrary, let $\big(U, (x^i)\big)$ be a chart around p, and write
$$\omega = \sum_i \omega_i\, dx^i.$$
For $1 \leq i \leq n$, denote $X_i = \partial/\partial x^i$. Then
$$X_i\big(\omega(X_j)\big) - X_j\big(\omega(X_i)\big) = X_i(\omega_j) - X_j(\omega_i) = \frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}.$$
On the other hand, note that by part (b) of *Exercise* 5, *Sheet* 11, we have $[X_i, X_j] = 0$. Therefore, by applying (iii) to $X = X_i$ and $Y = X_j$, we obtain
$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = X_i\big(\omega(X_j)\big) - X_j\big(\omega(X_i)\big) = \omega\big([X_i, X_j]\big) = 0,$$
so (ii) follows.

(ii) $\Rightarrow$ (i): Suppose that (ii) holds. Let $p \in M$ be arbitrary and let $\big(U, (x^i)\big)$ be the chart around p given by (ii). We have seen in the lecture notes (before *Proposition 8.20*) that

$$d\omega = \sum_{i<j} \underbrace{\left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right)}_{=0 \text{ by (ii)}} dx^i \wedge dx^j$$

and thus $d\omega = 0$ on U. In particular, $d\omega_p = 0$, and as $p \in M$ was arbitrary, we conclude that $d\omega = 0$ on M; in other words, ω is closed.

(b) We first deal with ω. Consider the function

$$f \colon \mathbb{R}^2 \to, \ (x, y) \mapsto \sin(xy)$$

and observe that $df = \omega$; in other words, ω is exact, and therefore ω is closed. (This can also be verified with a direct computation).

We now deal with η. We compute that

$$\begin{aligned} d\eta &= d\left(x\cos(xy)\right) \wedge dx + d\left(y\cos(xy)\right) \wedge dy \\ &= \big(\left(\cos(xy) - xy\sin(xy)\right) dx + \left(-x^2\sin(xy)\right) dy\big) \wedge dx + \\ &\quad + \big(-y^2\sin(xy)\, dx + \left(\cos(xy) - xy\sin(xy)\right) dy\big) \wedge dy \\ &= -x^2\sin(xy)\, dy \wedge dx - y^2\sin(xy)\, dx \wedge dy \\ &= (x^2 - y^2)\sin(xy)\, dx \wedge dy, \end{aligned}$$

which does not vanish identically, that is, η is not closed (see also (a)(ii)), and thus η cannot be exact either.