CH. 9 : INTEGRATION ON MANIFOLDS

We first give a crash course on mnfds with boundary. They play a central role in the theory of integration on mnfds, which will be briefly discussed afterwards; see pp. 135-144.

· DEF. 9.1 : The closed n-dim upper half-space $\mathbb{H}^n \subseteq \mathbb{R}^n$ is defined as

$$\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}.$$

The interior and the boundary of $\mathbb{H}^n$ as a subset of $\mathbb{R}^n$ are denoted by $\operatorname{Int} \mathbb{H}^n$ and $\partial \mathbb{H}^n$, respectively.

If $n > 0$, then

$$\operatorname{Int} \mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n > 0\},$$

$$\partial \mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n = 0\},$$

whereas if $n=0$, then

$$\mathbb{H}^0 = \mathbb{R}^0 = \{0\} \quad \text{and} \quad \partial \mathbb{H}^0 = \emptyset.$$

· DEF. 9.2 : An n-dim top. mnfd with boundary is a second-countable Hausdorff space M in which every pt has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $\mathbb{H}^n$.

An open subset $U \subseteq M$ together with a map $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ is called a chart for M. When it is necessary to make the distinction, we call (U, φ) an interior chart for M if $\varphi(U)$ is an open subset of $\mathbb{R}^n$ (which includes the case of an open subset of $\mathbb{H}^n$ that does not intersect $\partial\mathbb{H}^n$), and a boundary chart for M if $\varphi(U)$ is an open subset of $\mathbb{H}^n$ s.t. $\varphi(U) \cap \partial\mathbb{H}^n \neq \emptyset$.

A pt $p \in M$ is called an interior pt of M if it is in the domain of some interior chart, and a boundary pt of M if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$. The set of all boundary pts of M is denoted by ∂M and is called the boundary of M. The set of all interior pts of M is denoted by $\operatorname{Int} M$ and is called the interior of M.

· THM 9.3 (Topological invariance of the boundary): If M is a top. mnfd with boundary, then each pt of M is either a boundary pt or an interior pt, but not both. Thus, ∂M and $\operatorname{Int} M$ are disjoint sets whose union is M.

· EXAMPLE 9.4:

1) Every interval in $\mathbb{R}$ is a (connected) 1-mnfd with boundary, whose mnfd boundary consists of its endpts (if any).

2) The closed unit ball $\overline{\mathbb{B}}^n \subseteq \mathbb{R}^n$ is an n-mnfd with boundary, whose mnfd boundary is $\mathbb{S}^{n-1}$.

· PROP. 9.5: Let M be a top. n-mnfd with boundary.

(a) $\operatorname{Int} M$ is an open subset of M and a top. n-mnfd without boundary.

(b) ∂M is a closed subset of M and a top. $(n-1)$-mnfd without boundary.

(c) M is a top. mnfd (in the sense of DEF. 1.1) iff $\partial M = \emptyset$.

PROOF: Exercise! ■

Next, if U is an open subset of $\mathbb{H}^n$, then a map $F: U \to \mathbb{R}^k$ is said to be smooth if for each $x \in U$ there exists an open subset $\tilde{U} \subseteq \mathbb{R}^n$ containing x and a smooth map $\tilde{F}: \tilde{U} \to \mathbb{R}^k$ that agrees with F on $\tilde{U} \cap U$. If F is such a map, then the restriction of F to $U \cap \mathrm{Int}\,\mathbb{H}^n$ is smooth in the usual sense. By continuity, all partial derivatives of F at pts of $U \cap \mathrm{Int}\,\mathbb{H}^n$ are determined by their values in $\mathrm{Int}\,\mathbb{H}^n$, and thus in particular are independent of the choice of extension.

· DEF. 9.6: Let M be a top. mnfd with boundary. A smooth structure for M is defined to be a maximal smooth atlas (a collection of charts whose domains cover M and whose transition maps (and their inverses) are smooth in the sense just described). With such a structure, M is called a smooth mnfd with boundary.

In the following lengthy remark we collect some basic dfns and facts about smooth mnfds with boundary.

· REM. 9.7 :

1) cf. CH. 2 : Smoothness of a map $F: M \to N$ between mnfds with boundary is defined in the same way (see DEF. 2.3), with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each pt, and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$.

Smooth partitions of unity exist on smooth mnfds with boundary.

2) cf. CH. 3 : If M is a smooth n-mnfd with boundary, then the tangent space T_pM to M at $p \in M$ is defined in the same way (see DEF. 3.4), and it is an n-dim $\mathbb{R}$-v.s. For any smooth chart $(U, (x^i))$ containing p, the coordinate vectors

$$\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p$$

(where $\partial/\partial x^n|_p$ should be interpreted as a one-sided derivative when $p \in \partial M$) form a basis for T_pM.

The differential of a smooth map $F: M \to N$ between smooth mnfds with boundary is defined in the same way (see DEF. 3.6) and has the same representation in coordinates bases.

3) cf. CH. 4 : Submersions, immersions, embeddings and local diffeomorphisms are defined in the same way (see DEF. 4.2), and

there is a version of the rank theorem in this setting; see [Lee, Thm 4.15 and Problem 4.3].

4) cf. CH. 5 : Immersed and embedded submnfds of smooth mnfds with boundary are defined in the same way (see DEF. 5.1 and DEF. 5.12) and are themselves smooth mnfds with (possibly empty) boundary.

- ⇝ properties of embedded submnfds with boundary = = [Lee, Prop. 5.49]
- ⇝ version of the regular level set theorem in this setting = = [Lee, Problem 5.23]

THM: If M is a smooth n-mnfd with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-mnfd (without boundary), and has a unique smooth structure s.t. it is a properly embedded submnfd of M.

5) cf. CH. 7 : The tangent bundle of a smooth n-mnfd with boundary is defined in the same way (see DEF. 3.6) and it is a smooth vector bundle of rk n over the given mnfd (see PROP. 6.5). Vector fields are also defined in the same way (see DEF. 7.1), but flows need to be treated with extra care; see [Lee, Subsection 9.4]

Let M be a smooth mnfd with boundary and let $p \in \partial M$. It is intuitively evident that the vectors in T_pM can be separated in three classes: those tangent to the boundary, those pointing inward, and those pointing outward. Formally, we make the following

DEF.: If $p \in \partial M$, then a vector $v \in T_pM \setminus T_p\partial M$ is said to be inward-pointing if for some $\varepsilon > 0$ there exists a smooth curve $\gamma : [0, \varepsilon) \longrightarrow M$ s.t. $\gamma(0) = p$ and $\gamma'(0) = v$, and it is outward-pointing if there exists such a curve with domain $(-\varepsilon, 0]$.

PROP.: Let M be a smooth n-mnfd with boundary, $p \in \partial M$, and (x^i) be any smooth boundary coordinates defined on a neighborhood of p. The inward-pointing vectors in T_pM are precisely those with positive x^n-component, the outward-pointing ones are those with negative x^n-component, and the ones tangent to ∂M are those with zero x^n-component. Thus, T_pM is the disjoint union of $T_p\partial M$, the set of inward-pointing vectors, and the set of outward-pointing vectors. Finally, $v \in T_pM$ is inward-pointing iff $-v$ is outward-pointing.

The next result is used when discussing "boundary orientations"; see [Orientations, Subsection 2.3].

PROP.: If M is a smooth mnfd with boundary, then there exists a global smooth vector field on M whose restriction to ∂M is everywhere inward-pointing, and one whose restriction to ∂M is everywhere outward-pointing.

6) Cf. CH. 8: The cotangent bundle T^*M (resp. the k-th exterior power $\Lambda^k(T^*M)$ of the cotangent bundle) of a smooth n-mnfd M with boundary is defined in the same way (see DEF. 8.2 (resp.

DEF. 8.15(b))), and it is a smooth vector bundle of rk n (resp. of rk $\binom{n}{k}$) over M (see PROP. 8.3 (resp. p.118)). <u>Differential k-forms</u> $(0 \leq k \leq n)$ are also defined in the same way (see DEF. 8.15(b)), and so does their <u>exterior derivative</u> as well (see THM 8.21).

Orientations of manifolds, which also play an important role in integration theory on mnfds, are briefly discussed in the PDF [Orientations].

We are now ready to develop the general theory of integration on oriented mnfds. We first define the integral of a differential form over a domain in Euclidean space, and then we show how to use diffeomorphism invariance and smooth partitions of unity to extend this dfn to n-forms on oriented n-mnfds. The key feature of this dfn is that it is invariant under orientation-preserving diffeomorphisms. Afterwards, we state Stokes theorem (without proof), which is a generalization of the fundamental thm of calculus, and we also provide some applications; see pp. 142-144.

· <u>DEF. 9.8</u>: Let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and let ω be a compactly supported n-form on U. We define

$$\int_U \omega := \int_D \omega ,$$

where $D \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ is any domain of integration (e.g. a rectangle) containing supp ω, and ω is extended to be zero on the complement of its support.

Note that DEF. 9.8 does not depend on the choice of <u>domain of integration</u> (which is a bounded subset of $\mathbb{R}^n$ whose boundary has measure zero). Moreover, since ω can be written as $\omega = f\, dx^1 \wedge \dots \wedge dx^n$ for some cont fnct $f : \operatorname{supp}\omega \to \mathbb{R}$, the RHS in the above dfn is (the usual integral)

$$\int_D \omega = \int_D f\, dx^1 \dots dx^n .$$

<u>PROP. 9.9:</u> Let D and E be open domains of integration in $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G : \bar{D} \to \bar{E}$ be a smooth map that restricts to an orientation-preserving or orientation-reversing diffeomorphism $D \to E$. If ω is an n-form on $\bar{E}$, then

$$\int_D G^*\omega = \begin{cases} \int_E \omega , & \text{if } G \text{ is orient. pres} \\ -\int_E \omega , & \text{if } G \text{ is orient. reversing} \end{cases}$$

<u>PROOF:</u> Follows from the (usual) change of variables formula and the pullback formula for n-forms (PROP. 8.18). ∎

Since we cannot guarantee that arbitrary open or compact subsets are domains of integration, we need the following lemma in order to extend PROP. 9.9 to compactly supported n-forms defined on open subsets; see PROP. 9.11.

LEM 9.10: Let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and let K be a compact subset of U. There is an open domain of integration D s.t.

$$K \subseteq D \subseteq \bar{D} \subseteq U.$$

PROP. 9.11: Let U and V be open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G: U \to V$ be an orientation-preserving or orientation-reversing diffeomorphism. If ω is a compactly supported n-form on V, then

$$\int_V \omega = \begin{cases} \int_U G^*\omega, & \text{if } G \text{ is orient. preserving,} \\ -\int_U G^*\omega, & \text{if } G \text{ is orient. reversing.} \end{cases}$$

PROOF: By LEM 9.10 there is an open domain of integration E s.t. $\operatorname{supp}\omega \subseteq E \subseteq \bar{E} \subseteq V$. Since diffeomorphisms take interiors to interiors, boundaries to boundaries, and sets of measure zero to sets of measure zero, $D := G^{-1}(E) \subseteq U$ is an open domain of integration containing $\operatorname{supp} G^*\omega$. We conclude by PROP. 9.9. ∎

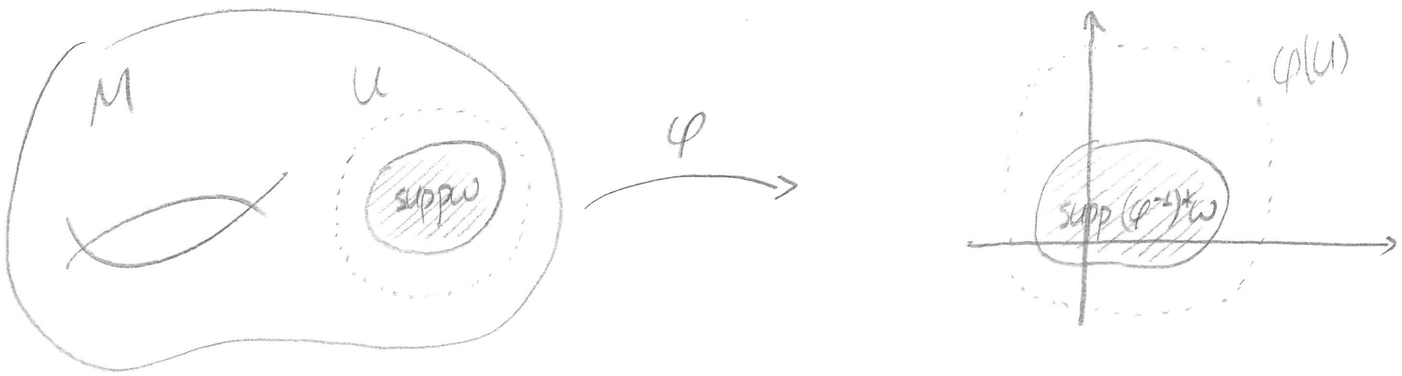
Using the above proposition we can now make sense of the integral of a differential form over an oriented mnfd.

Let M be an oriented smooth n-mnfd with or without boundary and let ω be an n-form on M. Suppose first that ω is compactly supported in the domain of a single smooth chart (U, φ) that is either positively or negatively oriented

We define <u>the integral of ω over M</u> to be

$$\int_M \omega := \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega \qquad (*_1)$$

with the positive sign for a positively oriented chart, and the negative sign otherwise. Since $(\varphi^{-1})^*\omega$ is a compactly supported n-form on the open subset $\varphi(U) \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, its integral is defined as in DEF. 9.8; see pp. 135-136.



PROP. 9.12: If M and ω are as above, then $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains $\operatorname{supp}\omega$.

PROOF: Let (U,φ) and $(\tilde{U},\tilde{\varphi})$ be two smooth charts s.t. $\operatorname{supp}\omega \subseteq U \cap \tilde{U}$. If both charts are similarly oriented, then $\tilde{\varphi}\circ\varphi^{-1} : \varphi(U\cap\tilde{U}) \to \tilde{\varphi}(U\cap\tilde{U})$ is an orientation-preserving diffeomorphism, see [Orientations, Ex. 25], so PROP. 9.11 yields

$$\int_{\tilde{\varphi}(\tilde{U})} (\tilde{\varphi}^{-1})^*\omega = \int_{\tilde{\varphi}(\tilde{U}\cap U)} (\tilde{\varphi}^{-1})^*\omega \overset{(9.11)}{=} \int_{\varphi(U\cap\tilde{U})} (\tilde{\varphi}\circ\varphi^{-1})^* (\tilde{\varphi}^{-1})^*\omega$$

$$= \int_{\varphi(U\cap\tilde{U})} (\varphi^{-1})^* \underbrace{\tilde{\varphi}^* (\tilde{\varphi}^{-1})^*\omega}_{(\tilde{\varphi}^{-1}\circ\tilde{\varphi})^* = \mathrm{Id}^*\omega} = \int_{\varphi(U)} (\varphi^{-1})^*\omega .$$

If the charts are oppositely oriented, then the two dfns given by $(*_1)$ have opposite signs, but this is compensated by the fact that $\tilde{\varphi} \circ \varphi^{-1}$ is orientation-reversing, so PROP. 9.11 introduces an extra negative sign into the above computation. ■

Let M be as above and let ω be a compactly supported n-form on M. Let $\{U_i\}$ be a finite open cover of $\operatorname{supp}\omega$ by domains of positively or negatively oriented smooth charts, and let $\{\psi_i\}$ be a smooth partition of unity subordinate to this covering. We define <u>the integral of ω over M</u> to be

$$\int_M \omega := \sum_i \int_M \psi_i \omega. \qquad (*_2)$$

Since for each i the n-form $\psi_i\omega$ is compactly supported in U_i, each of the terms in this (finite) sum is well defined according to our previous discussion. The following proposition shows that the integral is well defined.

· <u>PROP. 9.13</u>: The definition $(*_2)$ does not depend on the choice of open cover or partition of unity.

<u>PROOF</u>: Let $\{\tilde{U}_j\}$ be another finite open cover of $\operatorname{supp}\omega$ by domains of positively or negatively oriented smooth charts, and let $\{\tilde{\psi}_j\}$ be a subordinate smooth partition of unity. Since

$$\int_M \psi_i \omega = \int_M \Big(\sum \tilde{\psi}_j\Big)\psi_i\omega = \sum_j \int_M \tilde{\psi}_j \psi_i \omega, \ \forall i,$$

we obtain

$$\sum_i \int_M \psi_i \omega = \sum_{i,j} \int_M \tilde{\varphi}_j \psi_i \omega .$$

Each term in this sum is the integral of a form that is compactly supported in the domain of a single smooth chart (e.g. in U_i), so by PROP. 9.12 each term is well defined, regardless of which coordinate map we use to compute it. The same argument, starting with $\int_M \tilde{\varphi}_j \omega$ instead, shows that

$$\sum_j \int_M \tilde{\varphi}_j \omega = \sum_{i,j} \int_M \tilde{\varphi}_j \psi_i \omega .$$

Thus, both dfns yield the same value for $\int_M \omega$. ∎

If $S \subseteq M$ is an oriented immersed k-dim submnfd (with or without boundary) and ω is a k-form on M whose restriction to S is compactly supported, then we interpret $\int_S \omega$ as $\int_S \iota^* \omega$, where $\iota : S \hookrightarrow M$ is the inclusion map. In particular, if M is a compact, oriented, smooth n-mnfd with boundary and ω is an $(n-1)$-form on M, then we can interpret $\int_{\partial M} \omega$ unambiguously as the integral of $\iota_{\partial M}^* \omega$ over ∂M, where ∂M is always understood to have the induced orientation; see [Orientations, PROP 22].

· PROP. 9.14 (Properties of integrals): Let M and N be nonempty oriented smooth n-mnfds with or without boundary, and let ω and η be compactly supported n-forms on M.

(a) Linearity: If $a, b \in \mathbb{R}$, then
$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta .$$
(b) Orientation reversal: If $-M$ denotes M with the opposite orientation, then
$$\int_{-M} \omega = -\int_M \omega .$$
(c) Positivity: If ω is a positively oriented orientation form, then
$$\int_M \omega > 0 .$$
(d) Diffeomorphism invariance: If $F: N \to M$ is an orientation-preserving or orientation-reversing diffeomorphism, then
$$\int_M \omega = \begin{cases} \int_N F^*\omega, & \text{if } F \text{ is orient. preserving} \\ -\int_N F^*\omega, & \text{if } F \text{ is orient. reversing.} \end{cases}$$

PROOF:

(a) Exercise.

(b) Exercise (follows from the usual change of variables formula).

(c) Since ω is a positively oriented orientation form on M, if (U, φ) is a positively oriented smooth chart, then $(\varphi^{-1})^*\omega$ is a positive fnct times $dx^1 \wedge \dots \wedge dx^n$ (while if (U,φ) is negatively oriented, then it is a negative fnct times the same form); see the proof of [Orientations, PROP. 14]. Therefore, each term in $(*_2)$ defining $\int_M \omega$ is nonnegative, with at least one strictly positive term; this proves (c).

(d) It suffices to treat the case when ω is compactly supported in a single positively or negatively oriented smooth chart. If (U,φ) is a positively oriented such chart and if F is orientation preserving, then it is easy to check that $(F^{-1}(U), \varphi \circ F)$ is an oriented smooth chart on N whose domain contains $\operatorname{supp} F^*\omega$, so the result follows from PROP. 9.11. The remaining cases follow from this one and (b). ∎

We now state (without proof) the central result in the theory of integration on mnfds, Stokes' theorem. It is a far-reaching generalization of the fundamental thm of calculus and of the classical thms of vector calculus.

· THM 9.15 (Stokes thm). Let M be an oriented smooth n-mnfd with boundary and let ω be a compactly supported smooth $(n-1)$-form on M. Then

$$\int_M d\omega = \int_{\partial M} \omega .$$

Here, ∂M is understood to have the induced (Stokes) orientation, and the ω on the RHS is to be interpreted as $\iota_{\partial M}^* \omega$.

If $\partial M = \emptyset$, then the RHS is to be interpreted as 0. When M is 1-dim, the RHS is just a finite sum.

Finally, let us see some applications of Stokes' thm.

· <u>EXAMPLE 9.16</u>: Let M be a smooth mnfd. Let $\gamma : [a,b] \to M$ be a smooth embedding, so that $S := \gamma([a,b])$ is an embedded 1-submnfd with boundary in M. If we give S the orientation s.t. γ is orientation-preserving, then for any $f \in C^\infty(M)$, Stokes' thm says that

$$\int_\gamma df \overset{\text{E5.14}}{=} \int_{[a,b]} \gamma^* df \underset{(5.3)}{\overset{(9.14)}{=}} \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a))$$

(because $\partial S = \{\gamma(a), \gamma(b)\}$ is 0-dim). In particular, when $\gamma : [a,b] \to \mathbb{R}$ is the inclusion map, then Stokes' thm is just the ordinary fundamental theorem of calculus.

· <u>THM 9.17</u> (Green's thm): Let $D \subseteq \mathbb{R}^2$ be a compact regular domain (i.e., a properly embedded codim-0 submnfd with boundary), and let P, Q be smooth real-valued fncts on D. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{\partial D} P\,dx + Q\,dy.$$

<u>PROOF</u>: Apply Stokes' thm to the 1-form $P\,dx + Q\,dy$. ∎

· <u>COR. 9.18</u> (Integrals of exact forms): If M is a compact, oriented, smooth n-mnfd without boundary, then the integral of every exact n-form over M is zero:

$$\int_M d\omega = 0 \quad \text{if } \partial M = \emptyset.$$

· <u>COR. 9.19</u> (*Integrals of closed forms over boundaries*): Let M be a compact, oriented, smooth n-mnfd with boundary. If ω is a closed $(n-1)$-form on M, then the integral of ω over ∂M is zero:

$$\int_{\partial M} \omega = 0 \quad \text{if } d\omega = 0 \text{ on } M.$$

· <u>COR. 9.20</u>: Let M be a smooth mnfd with or without boundary, let $S \subseteq M$ be an oriented, compact, smooth k-dim submnfd (without boundary), and let ω be a closed k-form on M. If $\int_S \omega \neq 0$, then both of the following are true:

(a) ω is not exact on M.

(b) S is not the boundary of an oriented, compact, smooth submnfd with boundary in M.

<u>PROOF</u>:

(a) If ω were exact on M, then $\omega = d\eta$ for some $(k-1)$-form on M, so

$$0 \neq \int_S \omega \overset{\text{dfn}}{=} \int_S \iota_S^* \omega = \int_S \iota_S^* d\eta \overset{(8.22)}{=} \int_S d(\iota_S^* \eta) \overset{(9.18)}{=} 0 \ ↯.$$

(b) Argue again by contradiction and invoke (9.19). ■