Final Exam Solutions

Exercise 1. Quiz. (12 points)

For each statement below, tell whether it is true or false (1 pt), and provide a justification if the answer is "true" / a counter-example if the answer is "false" (2 pts).

a) (3 points) If a Markov chain is irreducible and recurrent, then it admits a stationary distribution.

Answer: False. Consider the symmetric random walk on $\mathbb{Z}$: it is irreducible and (null-)recurrent, but does not admit a stationary distribution.

b) (3 points) If a Markov chain admits a unique stationary distribution, then it is irreducible.

Answer: False. Consider a finite chain with two classes: a positive-recurrent one and a transient one (simplest example: $P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ admits the unique stationary distribution $\pi = (1, 0)$).

c) (3 points) Let P be the transition matrix of a finite and irreducible Markov chain with state space S, whose stationary distribution is uniform on S. Then $\sum_{i \in S} p_{ij} = 1$ for every $j \in S$.

Answer: True. Let $\pi_j = \frac{1}{N}$, $j \in S$, be the uniform stationary distribution on S (with $|S| = N$). Then $\pi\, P = \pi$, which translates into $\sum_{i \in S} p_{ij} = 1$ for every $j \in S$, after simplifying by $\frac{1}{N}$.

d) (3 points) Let P be the transition matrix of a finite and irreducible Markov chain, whose stationary distribution satisfies moreover detailed balance. If λ is an eigenvalue of P such that $|\lambda| = 1$, then $\lambda = +1$.

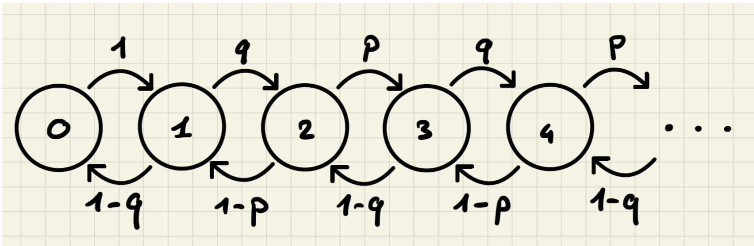
Answer: False. Consider an irreducible and 2-periodic chain (simplest example: $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$); then $\lambda = -1$ is an eigenvalue of P.

Exercise 2. (20 points)

Let $0 < p, q < 1$ and $(X_n,\, n \geq 0)$ be the time-homogeneous Markov chain with state space $S = \mathbb{N}$, transition matrix P given by

$$\begin{cases} p_{0,1} = 1, \quad p_{2k,2k+1} = p = 1 - p_{2k,2k-1} \quad \text{for } k \geq 1 \\ \\ p_{2k+1,2k+2} = q = 1 - p_{2k+1,2k} \quad \text{for } k \geq 0 \end{cases}$$

and the corresponding transition graph:



a) (6 points) Describe the set of values of $0 < p, q < 1$ for which the chain $(X_n,\, n \geq 0)$ admits a stationary distribution π and compute this stationary distribution.

Hint: Try detailed balance !

Answer: Solving the detailed balance equation gives (*NB:* when the transition matrix is tridiagonal, as it is in the present case, and a stationary distribution exists, it must be that detailed balance holds; therefore the hint):

$$\pi_1 = \frac{\pi_0}{1-q}, \quad \pi_2 = \frac{q}{(1-p)\,(1-q)}\,\pi_0, \quad \pi_{k+2} = \frac{pq}{(1-p)(1-q)}\,\pi_k, \quad k \geq 1$$

so it must be that

$$1 = \sum_{k \geq 0} \pi_k = \pi_0 + (\pi_1 + \pi_2) \sum_{k \geq 0} \left(\frac{pq}{(1-p)(1-q)} \right)^k$$

The sum on the right-hand side will converge if and only if $\left| \frac{pq}{(1-p)(1-q)} \right| < 1$, i.e., $p + q < 1$. In this case, we have

$$\begin{aligned} 1 &= \pi_0 \left(1 + \left(\frac{1}{1-q} + \frac{q}{(1-p)\,(1-q)} \right) \sum_{k \geq 0} \left(\frac{pq}{(1-p)(1-q)} \right)^k \right) \\ &= \pi_0 \left(1 + \frac{1-p+q}{(1-p)\,(1-q)} \, \frac{1}{1 - \frac{pq}{(1-p)(1-q)}} \right) = \pi_0 \left(1 + \frac{1-p+q}{1-p-q} \right) \\ &= \pi_0 \, \frac{2(1-p)}{1-p-q} \end{aligned}$$

so $\pi_0 = \frac{1-p-q}{2(1-p)}$ and the other values of π_k can be inferred from the equalities above.

b) (2 points) Under the condition found in part a), is the chain $(X_n,\, n \geq 0)$ ergodic ? Justify.

Answer: No, the chain is 2-periodic.

c) (3 points) For all values of $0 < p, q < 1$, compute $\mathbb{E}(T_0|X_0 = 0)$, where $T_0 = \inf\{n \geq 1 \,:\, X_n = 0\}$.

Answer: If $p + q \geq 1$, then $\mathbb{E}(T_0|X_0 = 0) = +\infty$, as in this case, the chain is either transient or null-recurrent. If $p + q < 1$, then

$$\mathbb{E}(T_0|X_0 = 0) = \frac{1}{\pi_0} = \frac{2(1-p)}{1-p-q}$$

d) (3 points) Let now $Q = P^2$. Explain in general why Q is guaranteed to be a transition matrix if P itself is a transition matrix.

Answer: Clearly, $q_{ij} \geq 0$ for every $i, j \in S$, and also:

$$\sum_{j \in S} q_{ij} = \sum_{j,k \in S} p_{ik}\, p_{kj} = \sum_{k \in S} p_{ik} \sum_{j \in S} p_{kj} = \sum_{k \in S} p_{ik} = 1$$

because P itself is a transition matrix (used twice here).

e) (3 points) Compute Q in the particular case of the present exercise.

Answer: We obtain

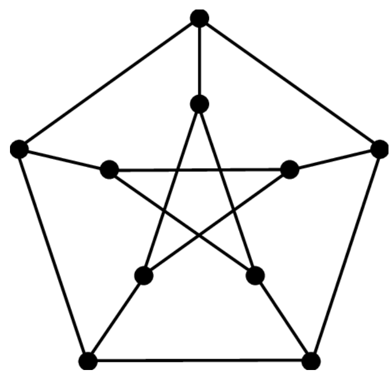
$$\begin{aligned}
&q_{00} = 1 - q, \quad q_{11} = 1 - pq, \quad q_{kk} = p + q - 2pq, \quad \text{for } k \geq 2 \\
&q_{02} = q, \quad q_{k,k+2} = pq, \quad \text{for } k \geq 1 \quad \text{and} \quad q_{k,k-2} = (1-p)\,(1-q), \quad \text{for } k \geq 2
\end{aligned}$$

f) (3 points) Under the condition found in part a), is the Markov chain $(Y_n,\, n \geq 0)$ with transition matrix Q ergodic ? Justify.

Answer: No, the chain is not irreducible (two classes $\{0, 2, 4, \ldots\}$ and $\{1, 3, 5, \ldots\}$).

Exercise 3. (16+3 points)

Consider the random walk on the Petersen (undirected) graph:



Let A be the adjacency matrix of this graph, defined as:

$$a_{ij} = \begin{cases} 1 & \text{if vertices } i \text{ and } j \text{ are connected by an edge} \\ 0 & \text{otherwise} \end{cases}$$

and let P be the transition matrix of the random walk on this graph, defined as

$$p_{ij} = \frac{a_{ij}}{d_i}, \quad \text{where } d_i \text{ is the degree of vertex } i$$

The aim of the present exercise is to compute the spectral gap of this random walk.

a) (2 points) Explain why this random walk is irreducible and aperiodic (and therefore ergodic, as it is finite).

Answer: Every state is reachable from every other state. And from a given state, there is a return path in either 2 or 5 steps, so the chain is also aperiodic, because $\gcd(2, 5) = 1$.

b) (2 points) From a given vertex i, determine the set of all vertices j which are reachable in two steps or less with this random walk.

Answer: From any vertex, all vertices of the graph are reachable in 2 steps or less.

c) (2 points) What is the stationary distribution π of the random walk ?

Answer: The graph is 3-regular, so the stationary distribution is uniform (doubly stochastic transition matrix).

BONUS d) (3 points) Show that $A^2 + A - 2I = J$, where I is the identity matrix and J is the "all ones" matrix, i.e., $J_{ij} = 1$ for all vertices i, j.

Answer: Note that $A_{ij} = 1$ if and only if $d(i, j) = 1$ and

$$(A^2)_{ij} = \begin{cases} 3 & \text{if } i = j \\ 0 & \text{if } d(i,j) = 1 \\ 1 & \text{if } d(i,j) = 2 \end{cases}$$

So indeed, $A^2 + A - 2I = J$ (remember that no two vertices are at mutual distance more than 2 (question b)).

e) (5 points) Using part d), deduce the set of possible values taken by the eigenvalues $\mu_0 \geq \mu_1 \geq \cdots \geq \mu_9$ of the adjacency matrix A.

Hint: The eigenvalues of the matrix J are given by

$$\nu_0 = 10, \quad \nu_1 = \nu_2 = \cdots = \nu_9 = 0$$

and watch out that only one eigenvalue μ_0 corresponds to the eigenvalue $\nu_0 = 10$.

Answer: The eigenvalue μ_0 corresponding to eigenvalue ν_0 must satisfy $\mu_0^2 + \mu_0 - 2 = 10$. Solving this equation for μ_0 gives $\mu_0 = +3$ or -4; it turns out (see below) that the value to retain is the value $+3$.

The other eigenvalues must satisfy $\mu^2 + \mu - 2 = 0$, i.e. $\mu = +1$ or -2.

(*NB:* This was not asked.) Besides, note that the total number of eigenvalues taking either value $+1$ or -2 is equal to 9, and that $\text{trace}(A) = 0 = 3 + (+1)\, n_{+1} + (-2)\, n_{-2}$, so $n_{+1} = 5$ and $n_{-2} = 4$.

f) (3 points) Using part e), deduce the set of possible values taken by the eigenvalues $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_9$ of the transition matrix P.

Answer: As P is a transition matrix, $\lambda_0 = +1$ is an eigenvalue, corresponding to the above eigenvalue $\mu_0 = +3$. The other eigenvalues must satisfy $\lambda = +\frac{1}{3}$ or $\lambda = -\frac{2}{3}$.

g) (2 points) Determine the value of the spectral gap γ of the random walk.

Answer: From f), we see that $\gamma = \frac{1}{3}$.

Exercise 4. (12 points)

Let $S = \mathbb{Z}^2$ and consider the following distribution on S:

$$\pi(i,j) = \frac{C}{1+i^2+j^2}, \quad (i,j) \in S$$

where $C > 0$ is the normalization constant such that $\sum_{(i,j)\in S} \pi(i,j) = 1$.

The aim of the present exercise is to sample from π using the Metropolis algorithm.

a) (4 points) Which of the following base chains on S are appropriate to start with ? Justify your answers.

Remarks:

- We do *not* ask here that the base chain is aperiodic.

- Each of proposed base chains below is represented by its sole transition matrix ψ.

- Some drawings are clearly recommended here !

a1) $\psi_{(i,j),(k,l)} = \begin{cases} 1/4 & \text{if } k=i+1,\, l=j+1 \\ 1/4 & \text{if } k=i+1,\, l=j-1 \\ 1/4 & \text{if } k=i-1,\, l=j+1 \\ 1/4 & \text{if } k=i-1,\, l=j-1 \\ 0 & \text{otherwise} \end{cases}$
a2) $\psi_{(i,j),(k,l)} = \begin{cases} 1/2 & \text{if } k=i+1,\, l=j \\ 1/8 & \text{if } k=i-1,\, l=j \\ 1/4 & \text{if } k=i,\, l=j+1 \\ 1/8 & \text{if } k=i,\, l=j-1 \\ 0 & \text{otherwise} \end{cases}$

a3) $\psi_{(i,j),(k,l)} = \begin{cases} 1/4 & \text{if } k=i+1,\, l=j+1 \\ 1/4 & \text{if } k=i+1,\, l=j \\ 1/4 & \text{if } k=i,\, l=j+1 \\ 1/4 & \text{if } k=i-1,\, l=j-1 \\ 0 & \text{otherwise} \end{cases}$
a4) $\psi_{(i,j),(k,l)} = \begin{cases} 1/4 & \text{if } k=i+1,\, l=j+1 \\ 1/4 & \text{if } k=i+1,\, l=j \\ 1/4 & \text{if } k=i-1,\, l=j \\ 1/4 & \text{if } k=i-1,\, l=j-1 \\ 0 & \text{otherwise} \end{cases}$

Answer: Base chains a2) and a4) are OK.

a1) is not irreducible

a3) does not satisfy the condition $\psi_{(i,j),(k,l)} > 0$ if and only if $\psi_{(k,l),(i,j)} > 0$.

b) (4 points) Consider now the base chain whose transition matrix is given by

$$\psi_{(i,j),(k,l)} = \begin{cases} 1/4 & \text{if } k = i+1,\, l = j \\ 1/4 & \text{if } k = i-1,\, l = j \\ 1/4 & \text{if } k = i,\, l = j+1 \\ 1/4 & \text{if } k = i,\, l = j-1 \\ 0 & \text{otherwise} \end{cases}$$

and compute the acceptance probabilities $a_{(i,j),(k,l)}$ of the Metropolis chain.

Remark: You may restrict yourselves to states in the first quadrant $\{(i,j) \in S : i \geq 0,\, j \geq 0\}$).

Answer: We obtain (for $i, j, k, l \geq 0$):

$$a_{(i,j),(k,l)} = \min\left\{1, \frac{\pi_{(k,l)}}{\pi_{(i,j)}}\right\} = \min\left\{1, \frac{1+i^2+j^2}{1+k^2+l^2}\right\}$$

$$= \begin{cases} 1 & \text{if } k = i,\, l = j-1 \quad \text{or} \quad k = i-1,\, l = j \\ \frac{1+i^2+j^2}{1+(i+1)^2+j^2} & \text{if } k = i+1,\, l = j \\ \frac{1+i^2+j^2}{1+i^2+(j+1)^2} & \text{if } k = i,\, l = j+1 \end{cases}$$

c) (4 points) Among all possible moves $(i,j) \to (k,l)$ proposed by the base chain ψ (that from part b), which have the least acceptance probability ? (multiple answers are possible, but only one is required)

Answer: Moves with the least acceptance probabilities are moves $(1,0) \to (2,0)$ and $(0,1) \to (0,2)$ (acceptance probability $= \frac{2}{5}$).