Astrophysics IV, Dr. Yves Revaz — EPFL

4th year physics — Exercises week 2

28.02.2024 — Spring semester 2024

Astrophysics IV : Stellar and galactic dynamics

Solutions

Problem 1 :

The disk surface density can be expressed as a function of the total mass and radius of the disk :

$$\Sigma = \frac{f M_{\rm tot}}{\pi R^2}. \quad (1)$$

Hence :

$$R = \sqrt{\frac{f \cdot M_{\rm tot}}{\pi \Sigma}}. \quad (2)$$

With $f = 1/45$, $M_{\rm tot} = 2 \cdot 10^{12}\,{\rm M}_\odot$, we thus have : $R = 16.8\,{\rm kpc}$.

The mean density is :

$$< \rho >= \frac{f \cdot M_{\rm tot}}{\pi R^2 \cdot 500} \sim 0.1\ {\rm M}_\odot\,{\rm pc}^{-3} \quad (3)$$

with a thickness of $500\,{\rm pc}$, $R = 16800\,{\rm pc}$ and $f = 1/45$.

The period of a circular orbit is :

$$T = \frac{2\pi R}{\sqrt{GM(R)/R}}, \quad (4)$$

So :

$$M = \frac{4\pi^2 R^3}{GT^2}. \quad (5)$$

With $G = 6.67 \cdot 10^{-11}\,{\rm m^3/kg/s^2}$, $R = 8\,{\rm kpc} = 2.46 \cdot 10^{20}\,{\rm m}$ and $T = 220\,{\rm Myr} = 6.9 \cdot 10^{15}\,{\rm s}$, we find approximately $10^{11}\,{\rm M}_\odot$.

Problem 2 :

For a galaxy cluster, the cluster typical radius is 1 Mpc and a typical galaxy size is 10 kpc :

$$\frac{\text{Volume (N galaxies)}}{\text{Volume (galaxy cluster)}} \simeq \frac{10^3 \cdot (10\ \text{kpc})^3}{(1000\ \text{kpc})^3} = 10^{-3}$$

For a galaxy :

$$\frac{\text{Volume (N stars)}}{\text{Volume (galaxy)}} \simeq \frac{10^{11} \cdot (10^6\ \text{km})^3}{(10^4\ \text{pc} \cdot 3.09 \cdot 10^{13}\ \text{km/pc})^3} \simeq 4 \times 10^{-24}$$

We thus see that those dynamical systems are largely made of void.

Problem 3 :

In a galaxy cluster, the typical speed of galaxies is $v = 10^3$ km/s$\cdot(3.09\cdot 10^{16}$ km/kpc$)^{-1} = 3.09 \cdot 10^{-14}$ kpc/s, hence :

$$\frac{\text{Volume (tube)}}{\text{Volume (galaxy cluster)}} \simeq \frac{\pi (10\text{ kpc})^2 \cdot v \cdot t}{\frac{4\pi}{3}(1000\text{kpc})^3} \simeq 7.7 \cdot 10^{-4}$$

Within a galaxy, stars typically have $v = 200$ km/s, hence :

$$\frac{\text{Volume (tube)}}{\text{Volume (galaxy)}} \simeq \frac{\pi (10^6\text{ km})^2 \cdot v \cdot t}{\frac{4\pi}{3}(10^4\text{ pc} \cdot 3 \cdot 10^{13}\text{ km/pc})^3} \simeq 1.6 \cdot 10^{-21}$$

Given the tiny portion of the volume crossed by the components, we conclude that the probability of a collision between two components of a given dynamical system is extremely low.

Problem 4 :

We consider that the gravitational influence of an object is significant when the mutual potential energy is of the same order than the kinematic energy of the relative motion.

$$E_{\text{cin}} \simeq E_{\text{grav}} \quad \Leftrightarrow \quad \frac{1}{2}mv^2 \simeq \frac{\text{G}m^2}{R_G} \quad \Leftrightarrow \quad R_G \simeq \frac{2\text{G}m}{v^2}$$

We observe that this result gives the value of $b_{90} = 2Gm/v^2$ seen in the course, so that $R_G = b_{90}$.

For a galaxy moving within a galaxy cluster :

$$R_G \simeq \frac{2 \cdot 6.67 \cdot 10^{-11}\text{m}^3/\text{kg/s}^2 \cdot 10^{11} \cdot 2 \cdot 10^{30}\text{ kg}}{(10^6\text{ m/s})^2} \simeq 10^{19}\text{ m} \simeq 1\text{ kpc}$$

For a star moving within a galaxy :

$$R_G \simeq \frac{2 \cdot 6.67 \cdot 10^{-11}\text{m}^3/\text{kg/s}^2 \cdot 2 \cdot 10^{30}\text{ kg}}{(2 \cdot 10^5\text{ m/s})^2} \simeq 0.7 \cdot 10^{10}\text{ m} \simeq 0.05\text{ A.U.}$$

Problem 5 :

The solution to this is given by the fraction of galaxies which have their symmetry axis (normal to the disk) which covers the ten degrees of the spherical cap.

$$\begin{aligned} P(i < i_0) &= \int_0^{i_0} \tfrac{2\pi \sin\theta}{2\pi} \mathrm{d}\theta \\ &= 1 - \cos i_0 \end{aligned}$$

For $i_0 = 10°$, we then get

$$P(i < 10°) = 1 - \cos 10° = \quad 0.015$$

For the fraction seen edge on, we have $P(i > 80°) = 1 - P(i < 80°) = 0.17$

Problem 6 :

The relaxation time is given by

$$t_{relax} = \frac{0.1\,N}{\ln N}\,t_{cross} = \frac{0.1\,N}{\ln N}\,\frac{R}{v}$$

Therefore we have

1. For the open cluster :

$$t_{relax} = \frac{0.1 \cdot 300}{\ln 300}\,\frac{2 \cdot 3.09 \cdot 10^{13}\ \mathrm{km}}{0.5\ \mathrm{km\,s^{-1}}} \simeq 6.5 \cdot 10^{14}\ \mathrm{s} \simeq 2 \cdot 10^{7}\ \mathrm{yrs}$$

   The youngest open clusters are not relaxed yet, but the oldest ones are fully relaxed.

2. For the globular cluster :

$$t_{relax} = \frac{0.1 \cdot 2 \cdot 10^{5}}{\ln(2 \cdot 10^{5})}\,\frac{3 \cdot 3.09 \cdot 10^{13}\ \mathrm{km}}{6\ \mathrm{km\,s^{-1}}} \simeq 2.5 \cdot 10^{16}\ \mathrm{s} \simeq 8 \cdot 10^{8}\ \mathrm{yrs}$$

   The globular clusters are fully relaxed.

3. For a dwarf spheroidal galaxy :

$$t_{relax} = \frac{0.1 \cdot 10^{7}}{\ln(10^{7})}\,\frac{500 \cdot 3.09 \cdot 10^{13}\ \mathrm{km}}{10\ \mathrm{km\,s^{-1}}} \simeq 1 \cdot 10^{20}\ \mathrm{s} \simeq 3 \cdot 10^{12}\ \mathrm{yrs}$$

   Dwarf spheroidal galaxies are far from being relaxed.

Problem 7 :

The relaxation time is given by a ratio between the velocity (which is a factor of the total mass of the system) and the average change in velocity per unit time. If the number of particles increases, the average velocity of a particle increases as $\propto \sqrt{N}$, while the average change in velocity per unit time only increases by $\propto \log N$. Therefore, if other factors are held constant, adding more members (and therefore mass) will result in an increase in the relaxation time of a system.