Quantum computation: lecture 4

- Communication complexity: classical setup
- Quantum communication complexity:
  - Yao's model
  - Cleve-Buhrman's model
- Distributed Deutsch-Josza's algorithm

Communication complexity

Alice $\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}$ Bob

Alice knows a vector $x \in \{0,1\}^n$

Bob knows a vector $y \in \{0,1\}^n$

They would like to compute together the value of $f(x,y)$, where $f: \{0,1\}^n \times \{0,1\}^n \to \mathbb{R}$ is some fn.

Def: communication complexity = minimum number of bits that Alice and Bob need to exchange in order to compute $f(x,y)$.

Example: $f(x,y) = DISJ(x,y)$

$= 1$ iff $\forall i = 1..n$, $x_i = 0$ or $y_i = 0$

$\Longrightarrow$ $\Omega(n)$ classical bits (i.e., at least order n bits) need to be exchanged in this case.

But with qubits, the situation is different...

1) Yao's model

Assume simply that Alice and Bob are allowed to exchange qubits. How many of them are needed?

Particular problem:

Let $d_H(x,y) = \#\{1 \le i \le n : x_i \ne y_i\}$ (= Hamming distance) and assume we know in advance that either $x = y$ (i.e. $d_H(x,y) = 0$) or $d_H(x,y) = \frac{n}{2}$

Classically, Alice & Bob need to exchange $\lceil \frac{n+1}{2} \rceil$ bits, in the worst case, in order to decide between these two alternatives.

We will see below that only $O(\log_2 n)$ qubits suffice.

Two remarks:

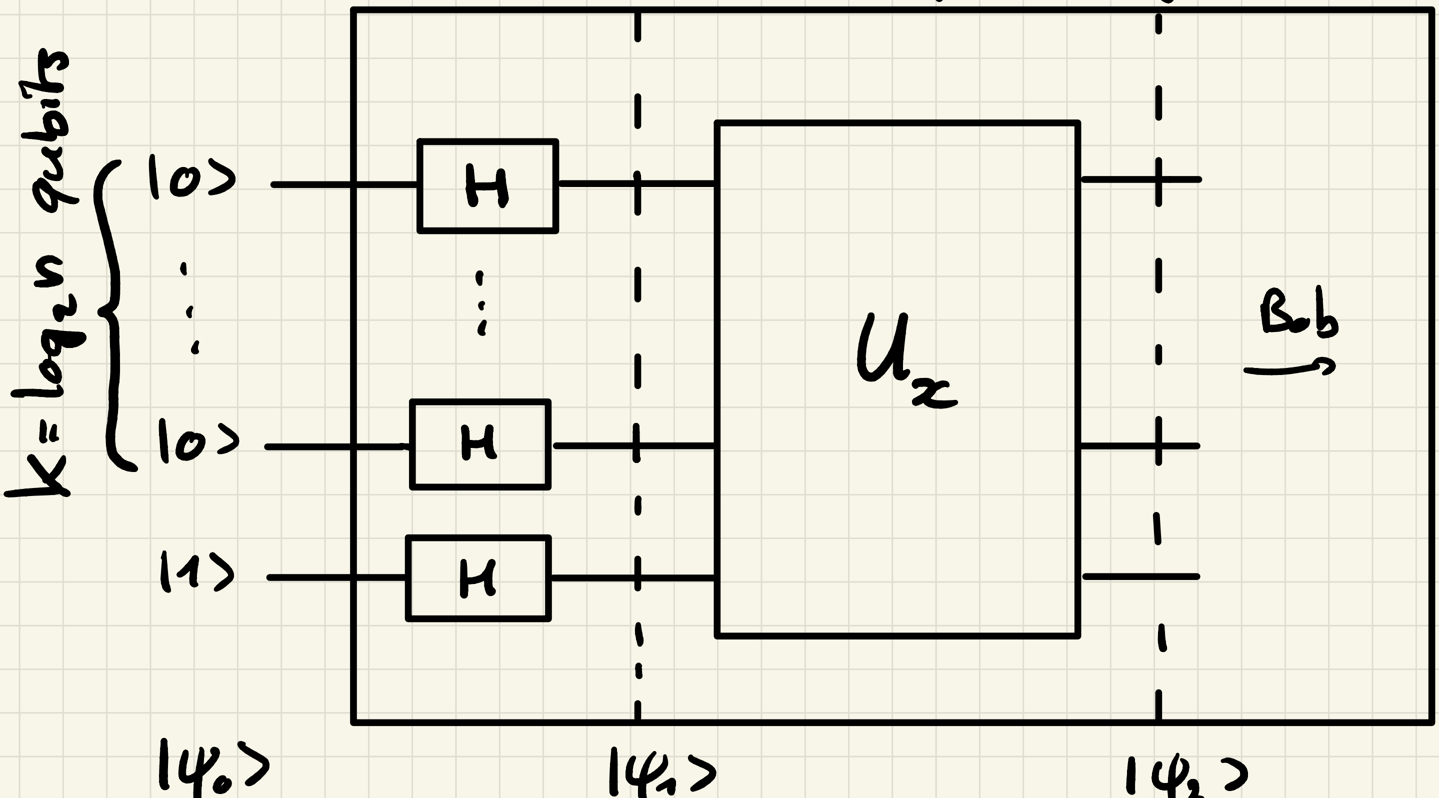
- For a reason that will become clear in a minute, we will write $\begin{cases} x = (x_0 \dots x_{n-1}) \\ y = (y_0 \dots y_{n-1}) \end{cases}$ instead of $\begin{cases} x = (x_1 \dots x_n) \\ y = (y_1 \dots y_n) \end{cases}$

- Observe that:

$$\begin{cases} x = y \quad \text{iff} \quad f(x,y) = \frac{1}{n} \sum_{i=0}^{n-1} (-1)^{x_i + y_i} = 1 \\ \\ d_H(x,y) = \frac{n}{2} \quad \text{iff} \quad f(x,y) = \frac{1}{n} \sum_{i=0}^{n-1} (-1)^{x_i + y_i} = 0 \end{cases}$$

Distributed Deutsch-Josza's algorithm

Alice designs first the following circuit:



NB: we assume $n = 2^k$, with k integer, for simplicity

Again, let us compute the states at the various stages:

- $$|\psi_0\rangle = \underbrace{|0\rangle \otimes \dots \otimes |0\rangle}_{k \text{ times}} \otimes |1\rangle = |0,\dots,0\rangle \otimes |1\rangle$$

- $$|\psi_1\rangle = H^{\otimes(k+1)} |\psi_0\rangle = H|0\rangle \otimes \dots \otimes H|0\rangle \otimes H|1\rangle$$
$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \dots \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$
$$= \frac{1}{2^{k/2}} \sum_{b_0 \dots b_{k-1} \in \{0,1\}} |b_0 \dots b_{k-1}\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$b_0 \dots b_{k-1}$ encodes a position $0 \leq b \leq 2^k - 1 = \underline{\underline{n-1}}$

$$\Longrightarrow |\psi_1\rangle = \frac{1}{\sqrt{n}} \sum_{0 \leq b \leq n-1} |b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

in short-hand notation

- <u>gate U_x</u>: its action on basis elements is given by:

$$U_x (|b\rangle \otimes |z\rangle) = |b\rangle \otimes |z \oplus x_b\rangle$$

So $U_x\left(|b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

$$= |b\rangle \otimes \frac{|x_b\rangle - |\overline{x_b}\rangle}{\sqrt{2}}$$

$$= \begin{cases} |b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} & \text{if } x_b = 0 \\ |b\rangle \otimes \frac{|1\rangle - |0\rangle}{\sqrt{2}} = -|b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} & \text{if } x_b = 1 \end{cases}$$

$$= (-1)^{x_b} \cdot |b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

(same magic as last time)

This gives

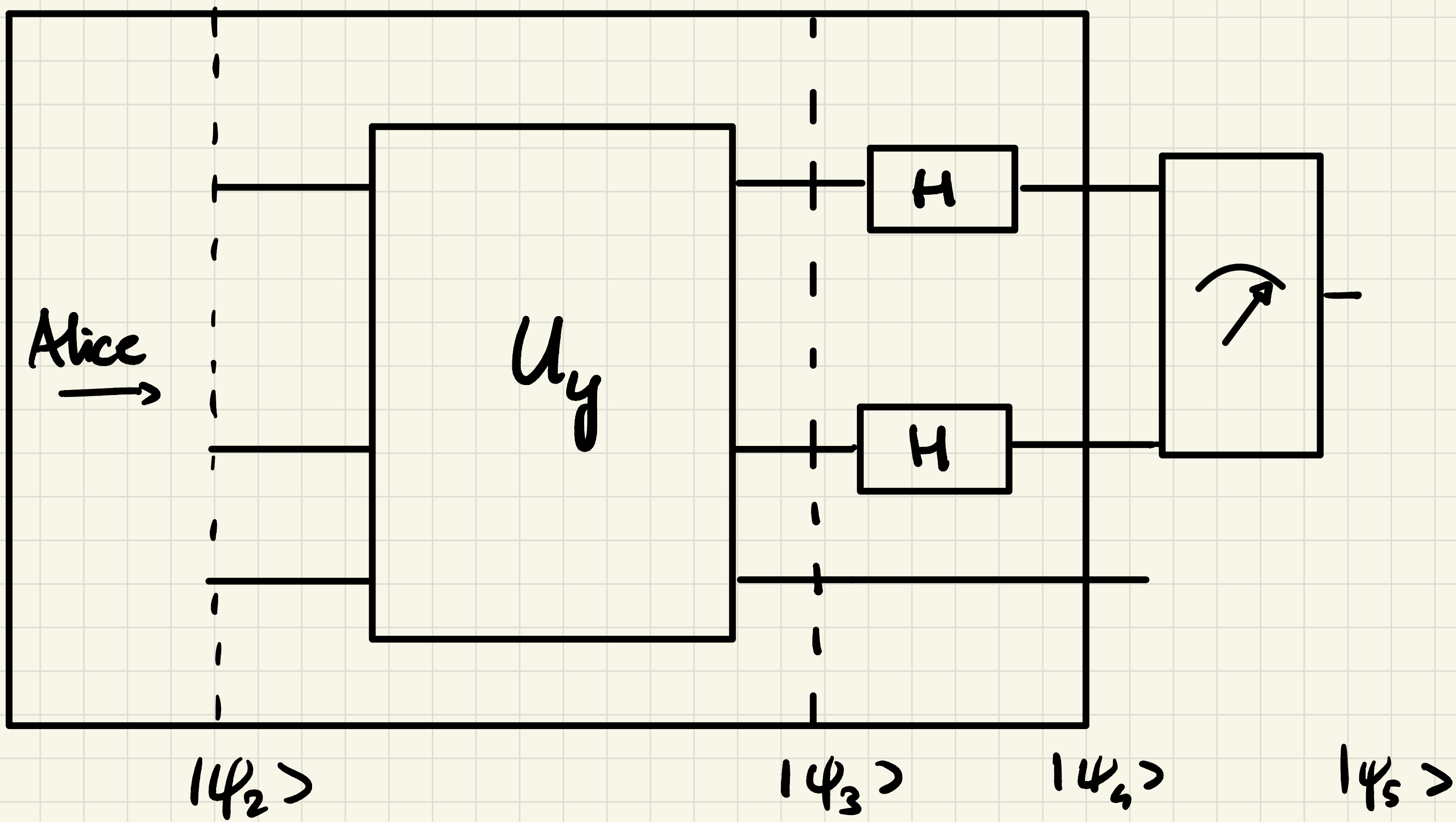
$$|\psi_2\rangle = U_x|\psi_1\rangle = \frac{1}{\sqrt{n}} \sum_{0\leq b\leq n-1} U_f\left(|b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{n}} \sum_{0\leq b\leq n-1} (-1)^{x_b} \cdot |b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Then Alice transmits this state $|\psi_2\rangle$ to Bob:

this amounts to transmitting $\log_2 n\ (+1)$ qubits.

Then Bob uses the following circuit:



- The action of the gate U_y is:

$$U_y(|b\rangle \otimes |z\rangle) = |b\rangle \otimes |z \oplus y_b\rangle$$

so $U_y\left(|b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = (-1)^{y_b} \cdot |b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

(same computation as before)

and

$$|\psi_3\rangle = U_y|\psi_2\rangle = \frac{1}{\sqrt{n}} \sum_{0 \leq b \leq n-1} (-1)^{x_b + y_b} \cdot |b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

• Finally : $|\psi_4\rangle = \left(H^{\otimes k} \otimes I\right)|\psi_3\rangle$

$$= \frac{1}{\sqrt{n}} \sum_{0 \le b \le n-1} (-1)^{x_b + y_b} \, H^{\otimes k}|b\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

where

$$H^{\otimes k}|b\rangle = H|b_0\rangle \otimes \dots \otimes H|b_{k-1}\rangle$$

and $\quad H|b_j\rangle = \frac{1}{\sqrt{2}} \sum_{c_j \in \{0,1\}} (-1)^{b_j c_j} |c_j\rangle$

so $H^{\otimes k}|b\rangle = \frac{1}{2^{k/2}} \sum_{c_0 \dots c_{k-1} \in \{0,1\}} (-1)^{b_0 c_0 + \dots + b_{k-1} c_{k-1}} |c_0 \dots c_{k-1}\rangle$

In short-hand notation:

$$H^{\otimes k} |b\rangle = \frac{1}{\sqrt{n}} \sum_{0 \leq c \leq n-1} (-1)^{b \cdot c} |c\rangle$$

where $b \cdot c := b_0 c_0 + b_1 c_1 + \ldots + b_{k-1} c_{k-1}$. Then

$$|\varphi_5\rangle = \sum_{0 \leq c \leq n-1} \underbrace{\left(\frac{1}{n} \sum_{0 \leq b \leq n-1} (-1)^{x_b + y_b + b \cdot c} \right)}_{= \alpha_c} |c\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

When measuring the first k qubits of $|\psi_4\rangle$, Bob obtains state $|c\rangle$ with probability $|\alpha_c|^2 = \left| \frac{1}{n} \sum_{0 \le b \le n-1} (-1)^{x_b + y_b + b \cdot c} \right|^2$

For $|c\rangle = |0 \ldots 0\rangle$, we obtain:

$$|\alpha_0|^2 = \left| \frac{1}{n} \sum_{0 \le b \le n-1} (-1)^{x_b + y_b} \right|^2$$

$$\begin{cases} \bullet \text{ If } x=y, \text{ then } |\alpha_0|^2 = 1 \\ \bullet \text{ If } d_H(x,y) = \frac{n}{2}, \text{ then } |\alpha_0|^2 = 0 \end{cases}$$

(cf remark a few pages backwards)

So Bob concludes that $x=y$ if he observes the state 0 and that $d_H(x,y)=\frac{n}{2}$ otherwise (and he can transmit this one-bit info to Alice).

And recall that only $k=\log_2 n$ qubits have been exchanged.

2) Cleve-Buhrman's model

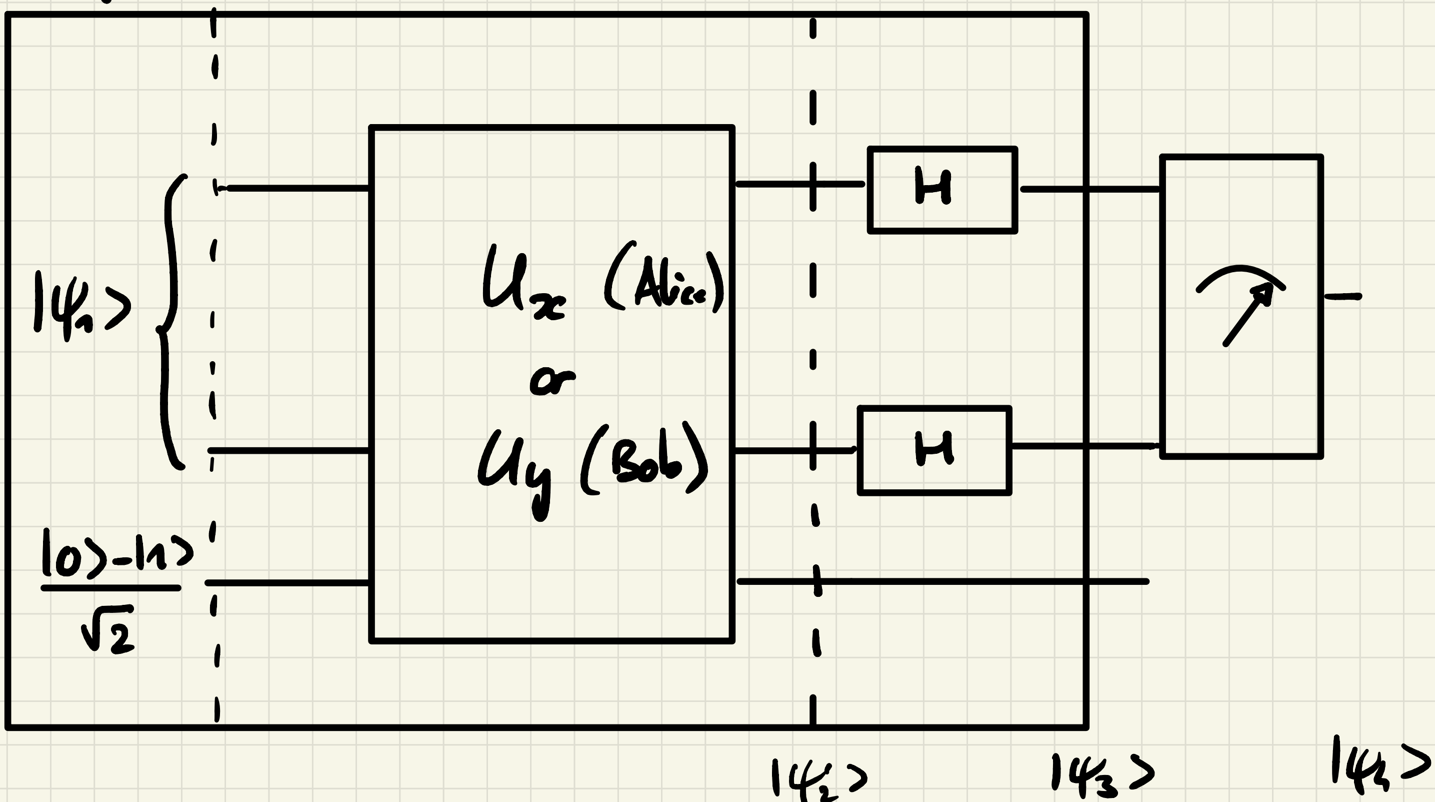
Still for the same problem (i.e. distinguishing between $x = y$ and $d_H(x,y) = \frac{n}{2}$), we suppose now that Alice & Bob own each $k = \log_2 n$ qubits which are entangled at the start:

$$|\varphi_1\rangle = \frac{1}{2^{k/2}} \sum_{b_0..b_{k-1} \in \{0,1\}} |b_0..b_{k-1}\rangle_A \otimes |b_0 \dots b_{k-1}\rangle_B$$

(<u>NB</u>: The previous state is nothing but the tensor product of k Bell states!)

The question is now: how many <u>classical bits</u> need Alice & Bob exchange in order to decide between the two alternatives?

Alice and Bob use separately the following circuit:



$$|\psi_2\rangle = \frac{1}{\sqrt{n}} \sum_{0\leq b\leq n-1} U_x\left(|b\rangle_A \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \otimes U_y\left(|b\rangle_B \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

ancilla bit of Alice

ancilla bit of Bob

$$= \frac{1}{\sqrt{n}} \sum_{0\leq b\leq n-1} (-1)^{x_b+y_b} |b\rangle_A \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}} \otimes |b\rangle_B \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

After the passage through the Hadamard gates:

(forget the ancilla bits)

$$|\psi_3\rangle = \frac{1}{\sqrt{n}} \sum_{0\leq b\leq n-1} (-1)^{x_b+y_b} H^{\otimes k}|b\rangle_A \otimes H^{\otimes k}|b\rangle_B$$

As before, $H^{\otimes k}|b\rangle_A = \frac{1}{\sqrt{n}} \sum_{0 \le c \le n-1} (-1)^{b \cdot c} |c\rangle_A$

and $H^{\otimes k}|b\rangle_B = \frac{1}{\sqrt{n}} \sum_{0 \le d \le n-1} (-1)^{b \cdot d} |d\rangle_B$, so

$$|\psi_3\rangle = \sum_{\substack{0 \le c \le n-1 \\ 0 \le d \le n-1}} \underbrace{\left(\frac{1}{n^{3/2}} \sum_{0 \le b \le n-1} (-1)^{x_b + y_b + b \cdot c + b \cdot d} \right)}_{= \alpha_{c,d}} |c\rangle_A \otimes |d\rangle_B$$

So after the measurement on both sides, the joint probability that Alice sees $|c\rangle_A$ and Bob sees $|d\rangle_B$ is $|\alpha_{c,d}|^2$.

And $|\alpha_{c,d}|^2 = \frac{1}{n^3} \left| \sum_{0 \leq b \leq n-1} (-1)^{x_b + y_b + b \cdot c + b \cdot d} \right|^2$

In particular:

$$|\alpha_{c,c}|^2 = \frac{1}{n^3} \left| \sum_{0 \leq b \leq n-1} (-1)^{x_b + y_b} \right|^2$$

So the probability that Alice & Bob dosenve the same state is given by

$$\sum_{c=0}^{n-1} |\alpha_{c,c}|^2 = \frac{1}{n^2} \left| \sum_{0 \leq b \leq n-1} (-1)^{x_b + y_b} \right| = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } d_H(x,y) = n/2 \end{cases}$$

So after having performed both their measurements, Alice (e.g.) sends to Bob $k = \log_2 n$ classical bits describing her observed state $|c\rangle$. If this state is equal to $|d\rangle$, then $x = y$; otherwise, this means $d_H(x,y) = \frac{n}{2}$.