Potential Theory

end of the 2nd part

Outlines

Axisymmetric models for disk galaxies

- “Potential based” models
- Potential of flattened systems
- Potential of infinite thin (razor-thin) disks
- “Potential based” razor-thin disks models
- Potential of spheroidal shells (homoeoids)
- Potential of spheroids
- Potential of infinite thin (razor-thin) disks from homoeoids

Ideal but useful models

- the infinite wire, the infinite slab
- infinite slab with oscillatory surface density, tightly wound spiral

Orbits

- some generalities

Axisymmetric models for disk galaxies

$$\rho(\vec{x}) = \rho(R, |z|)$$

$$R = \sqrt{x^2 + y^2}$$

Examples of axisymmetric models

“Potential based” models

Kuzmin disk

Kuzmin 1956

$$\Phi_{\mathrm{K}}(R,z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}} = -\frac{GM}{\sqrt{R^2 + z^2 + a^2 + 2\,a\,|z|}}$$

Comparison with Plummer:

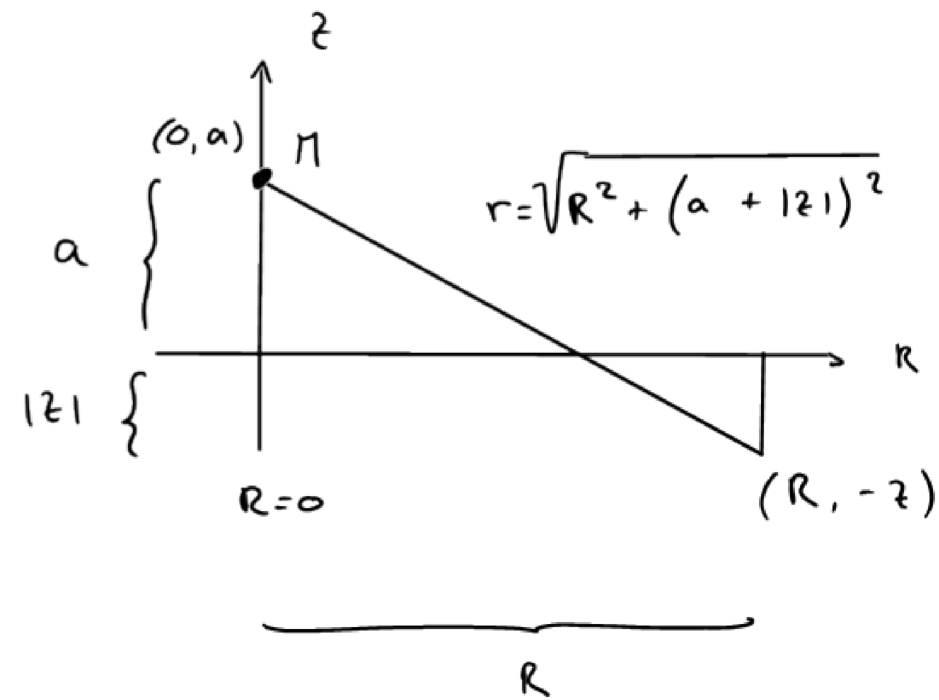
$$\Phi_{\mathrm{P}}(R,z) = -\frac{GM}{\sqrt{R^2 + z^2 + a^2}}$$

Equivalent to the following configuration

Potential due to

a mass M at $(0, a)$

$$\Rightarrow \quad -\frac{GM}{r} = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}}$$



Kuzmin disk

Kuzmin 1956

$$\Phi_{\rm K}(R,z) = -\frac{GM}{\sqrt{R^2 + (a + |z|)^2}}$$

Plummer based model

$$\Sigma_{\rm K}(R) = \frac{aM}{2\pi(R^2 + a^2)^{3/2}}$$

Infinitely thin disk

$$V_{c,{\rm K}}^2(R) = \frac{G\,M\,R^2}{\left(R^2 + a^2\right)^{3/2}}$$

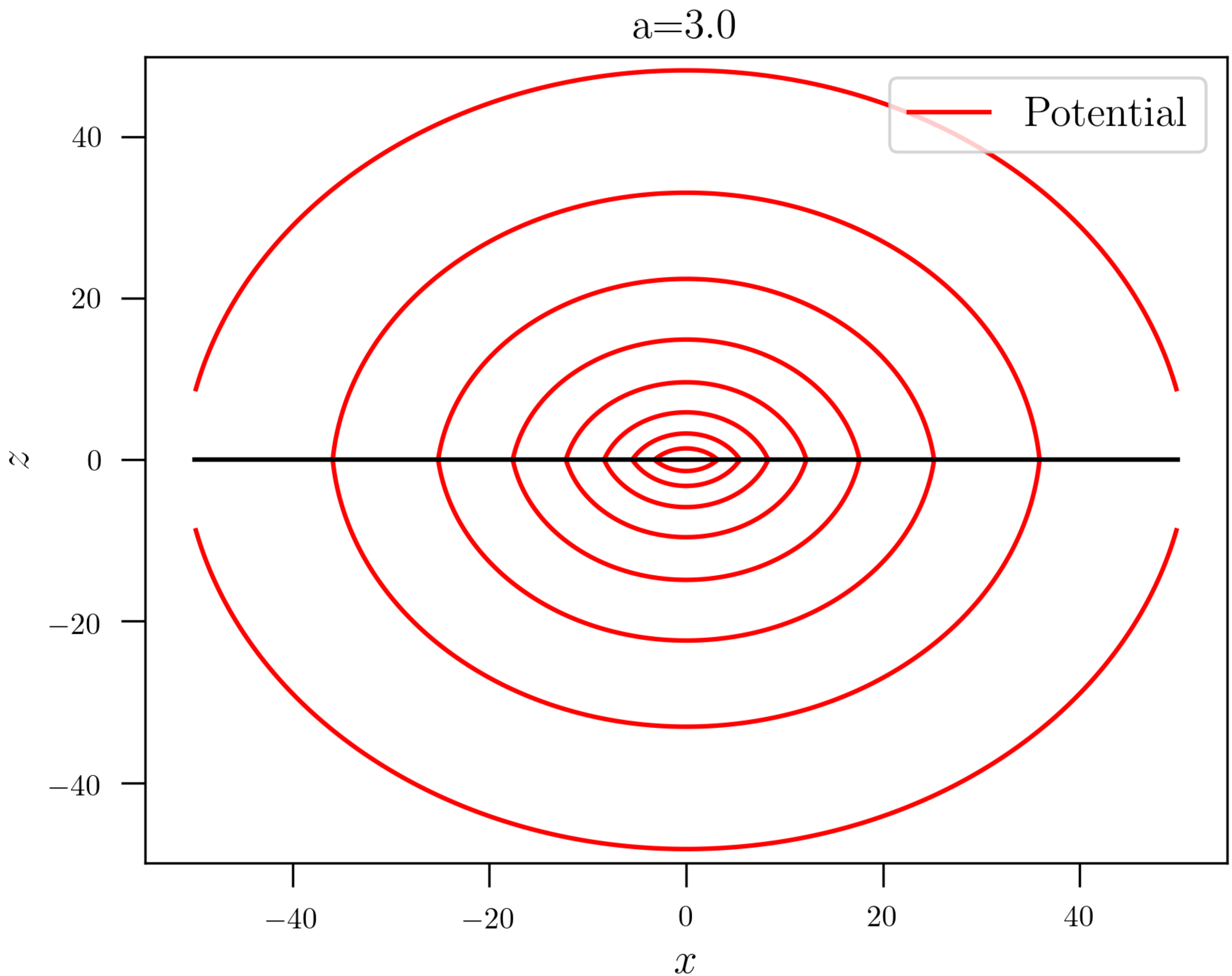
Equivalent to the Plummer model

$$V_{c,{\rm P}}^2(r) = \frac{GMr^2}{(r^2 + b^2)^{3/2}}$$

Note: for an axi-symmetric model, the circular velocity is computed in the plane z=0.

$$V_c^2(R) = \frac{1}{R}\frac{\mathrm{d}\Phi(\mathrm{R},\mathrm{z}=0)}{\mathrm{dR}}$$

Kuzmin disk



Miyamoto-Nagai potential

Miyamoto & Nagai 1975

$$\Phi_{\rm MN}(R,z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2+b^2})^2}}$$

b=0 → Kuzmin

$$\rho_{\rm MN}(R,z) = \left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a + 3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{[R^2 + (a+\sqrt{z^2+b^2})^2]^{5/2}(z^2+b^2)^{3/2}}$$

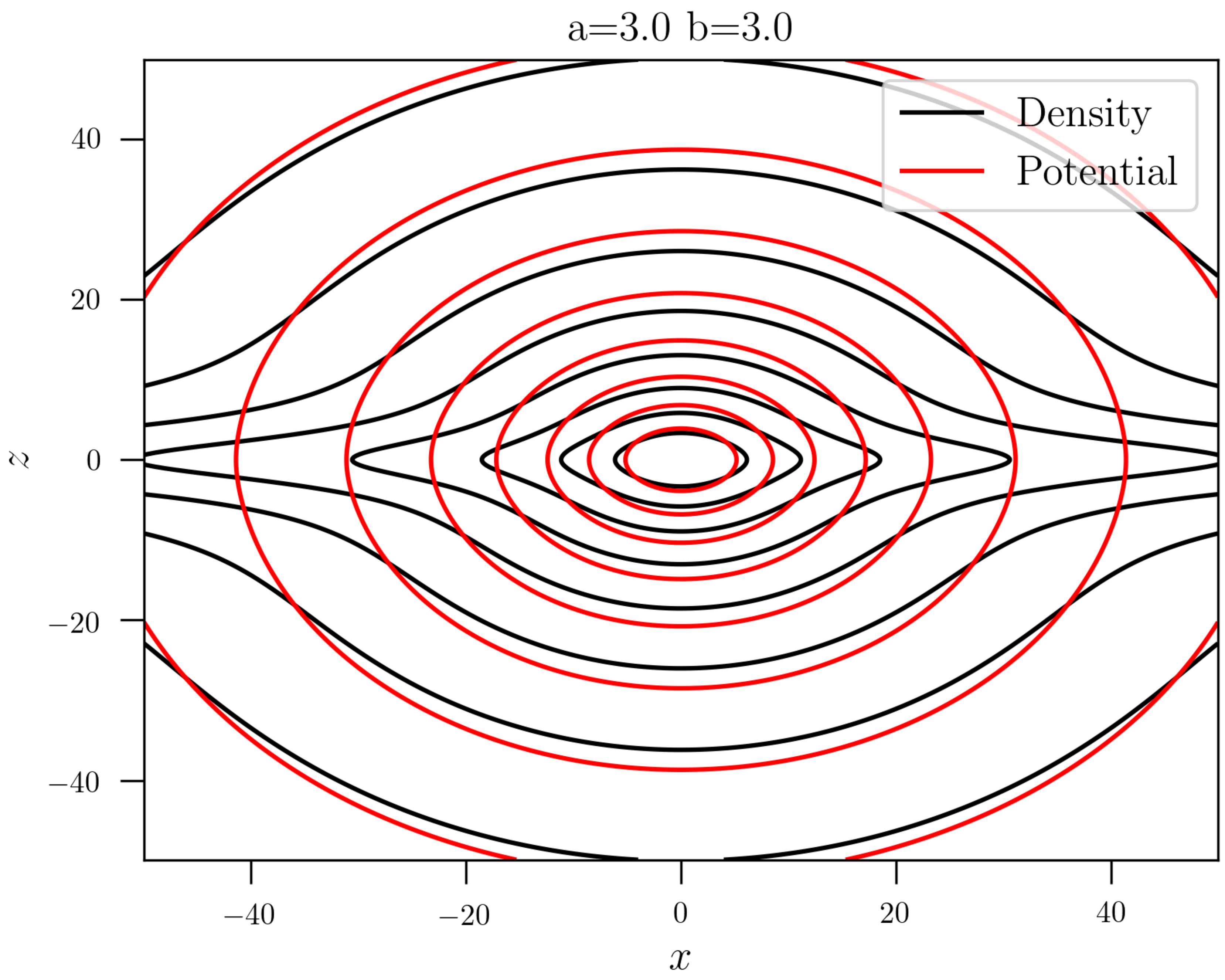
$$V_{c,\rm MN}^2(R) = \frac{G\,M\,R^2}{\left(R^2 + (a+b)^2\right)^{3/2}}$$

Equivalent to the Plummer model

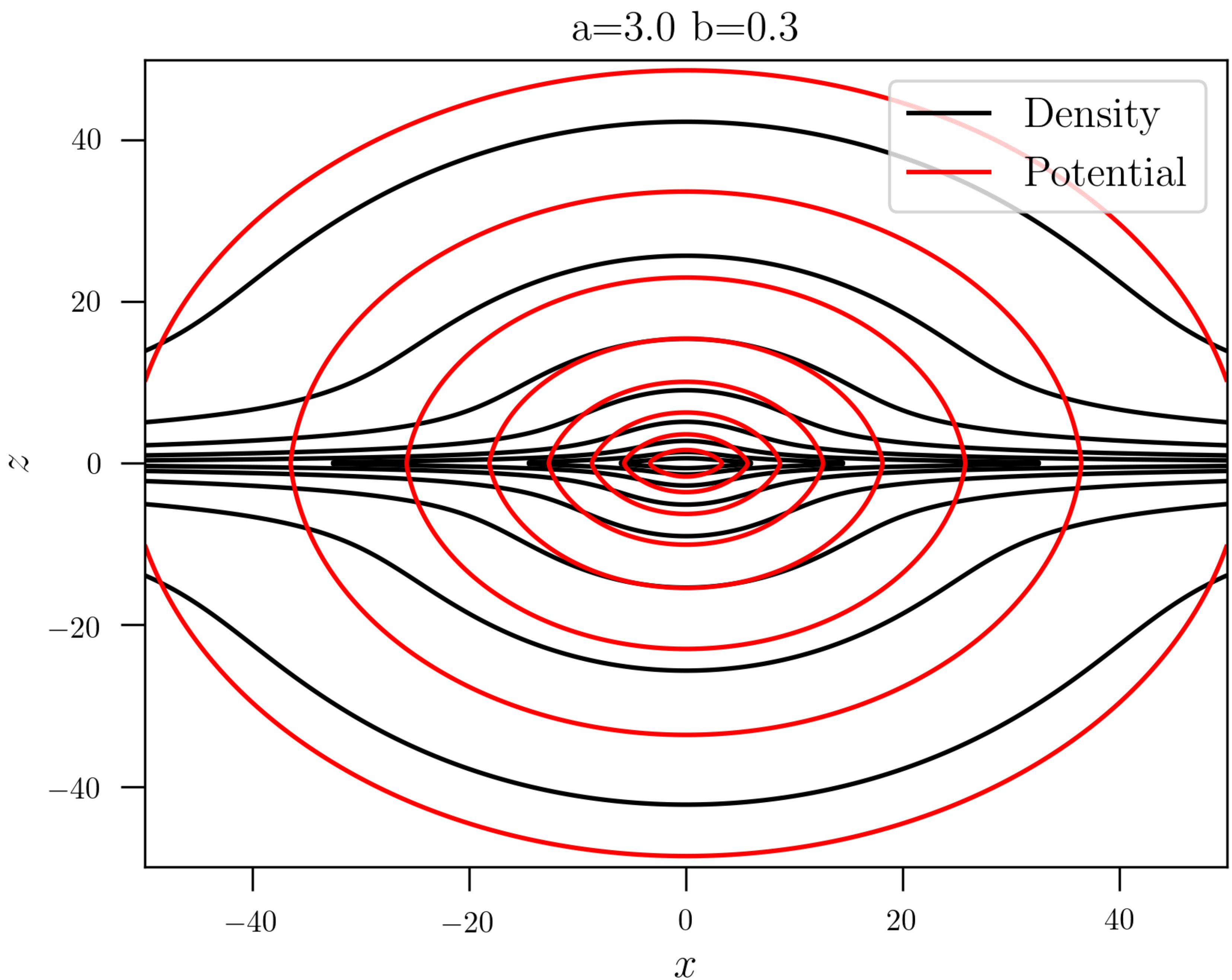
$$V_{c,\rm P}^2(r) = \frac{GMr^2}{(r^2+b^2)^{3/2}}$$

Better parametrisation :
Revaz & Pfenniger 2004

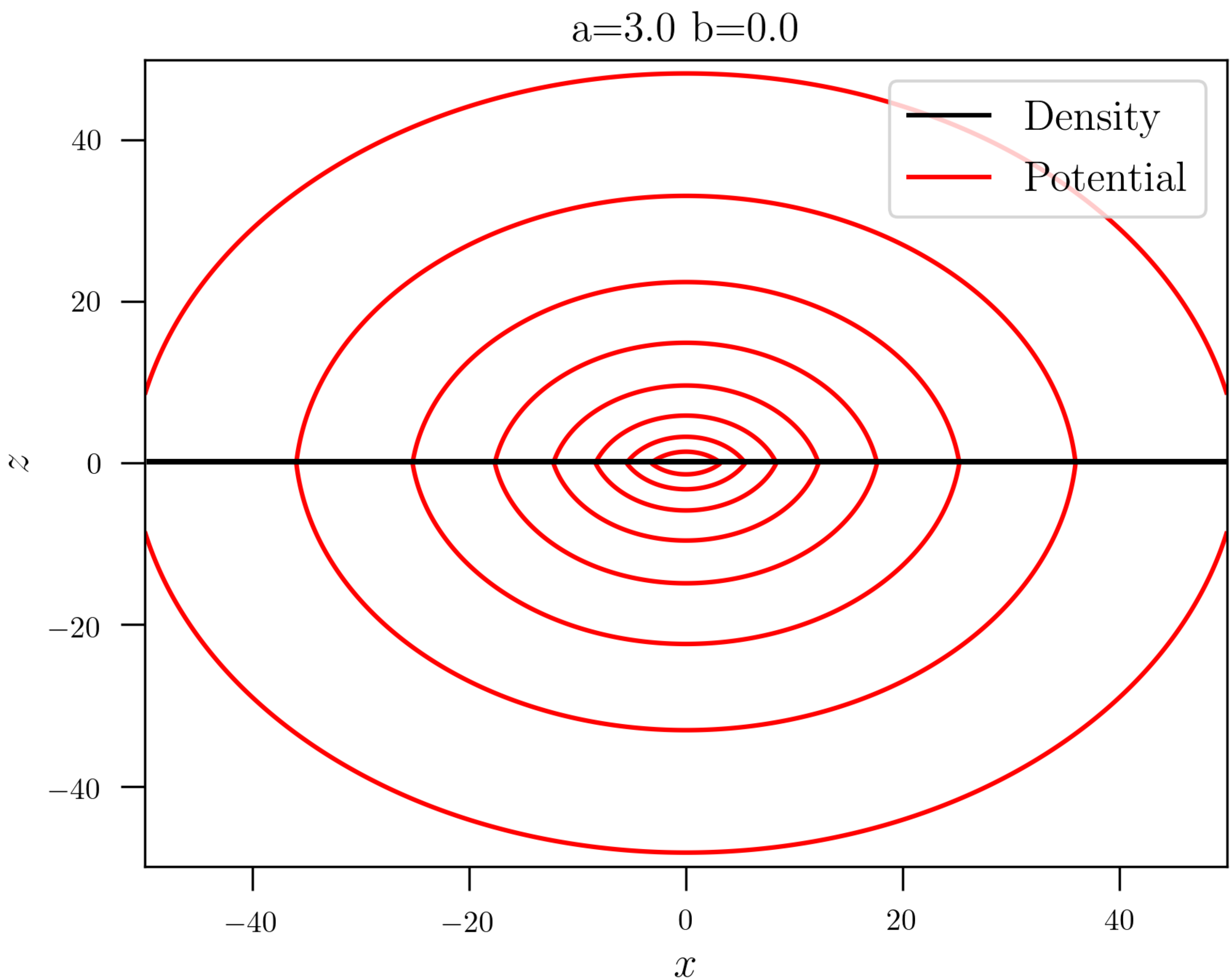
Miyamoto-Nagai potential



Miyamoto-Nagai potential



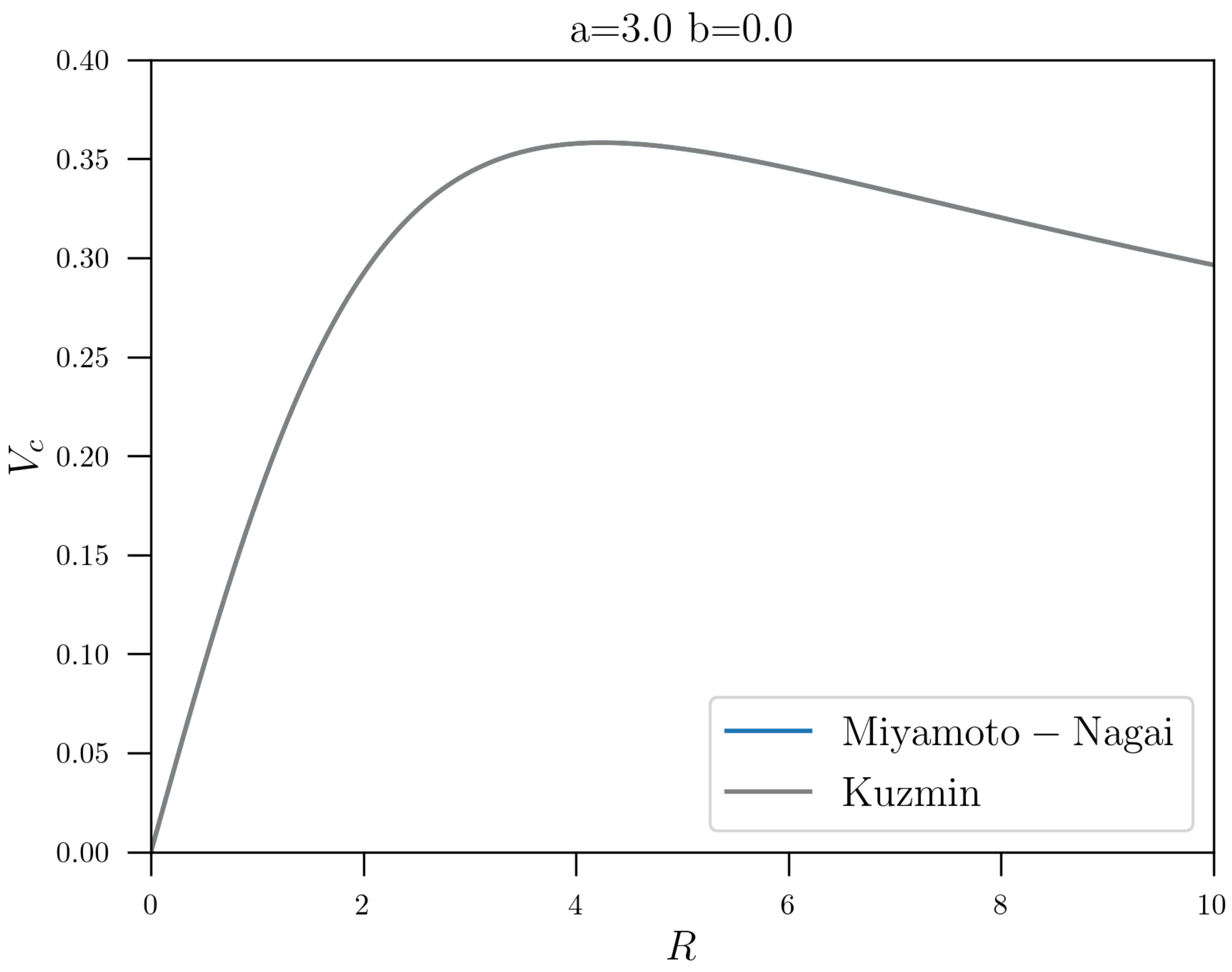
Miyamoto-Nagai potential



Miyamoto-Nagai potential

Miyamoto & Nagai 1975

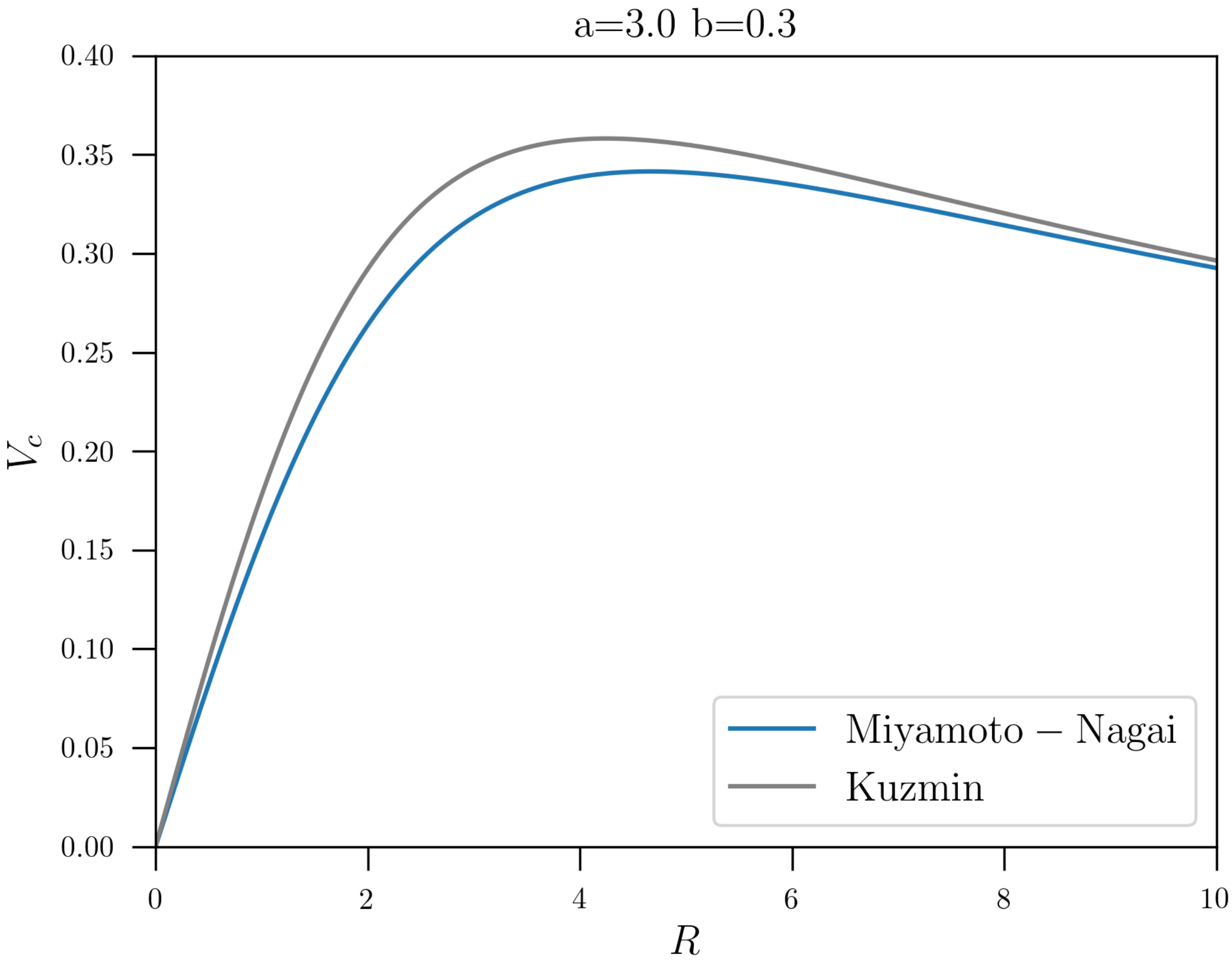
Circular velocity rotation curve



Miyamoto-Nagai potential

Miyamoto & Nagai 1975

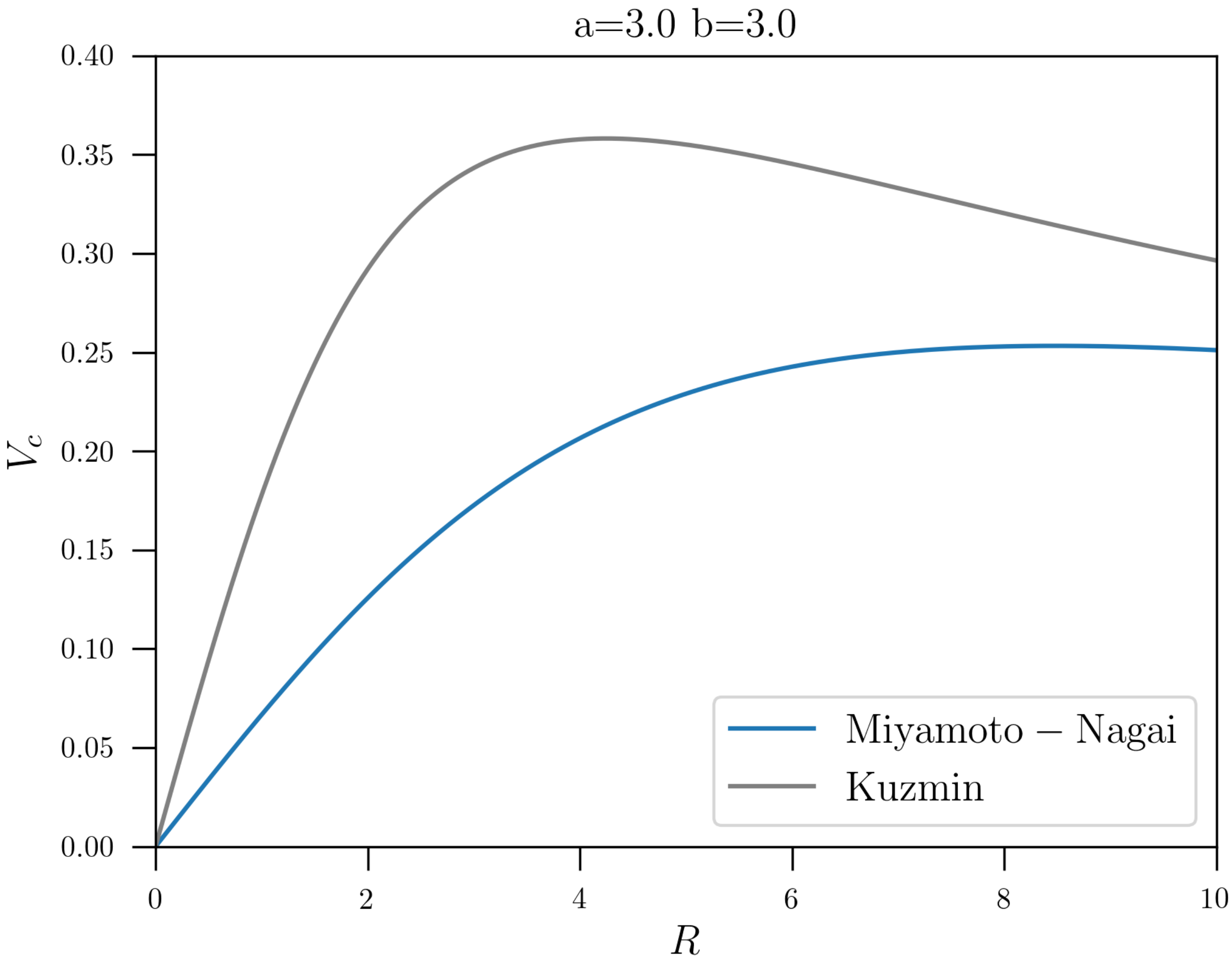
Circular velocity rotation curve



Miyamoto-Nagai potential

Miyamoto & Nagai 1975

Circular velocity rotation curve



Logarithmic potential

$$\Phi_{\log}(R,z) = \frac{1}{2} V_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{q^2}\right)$$

Rc=0 and q=1
→ Isothermal sphere

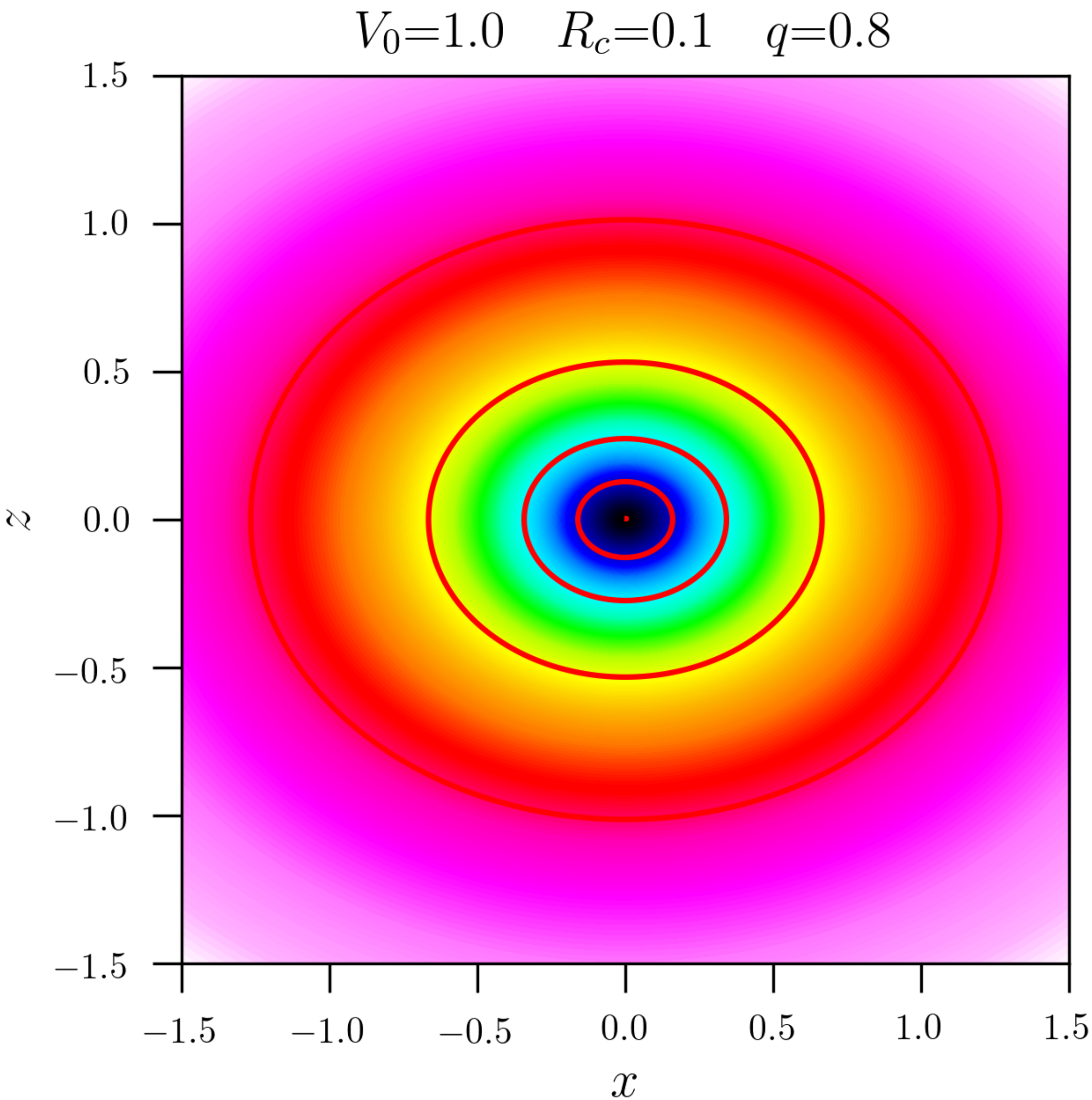
$$\rho_{\log}(R,z) = \frac{V_0^2}{4\pi G q^2} \frac{(2q^2+1)R_c^2 + R^2 + (2 - 1/q^2)z^2}{\left(R_c^2 + R^2 + (z^2/q^2)\right)^2}$$

- negative for $q < 1\sqrt{(2)} \cong 0.707$

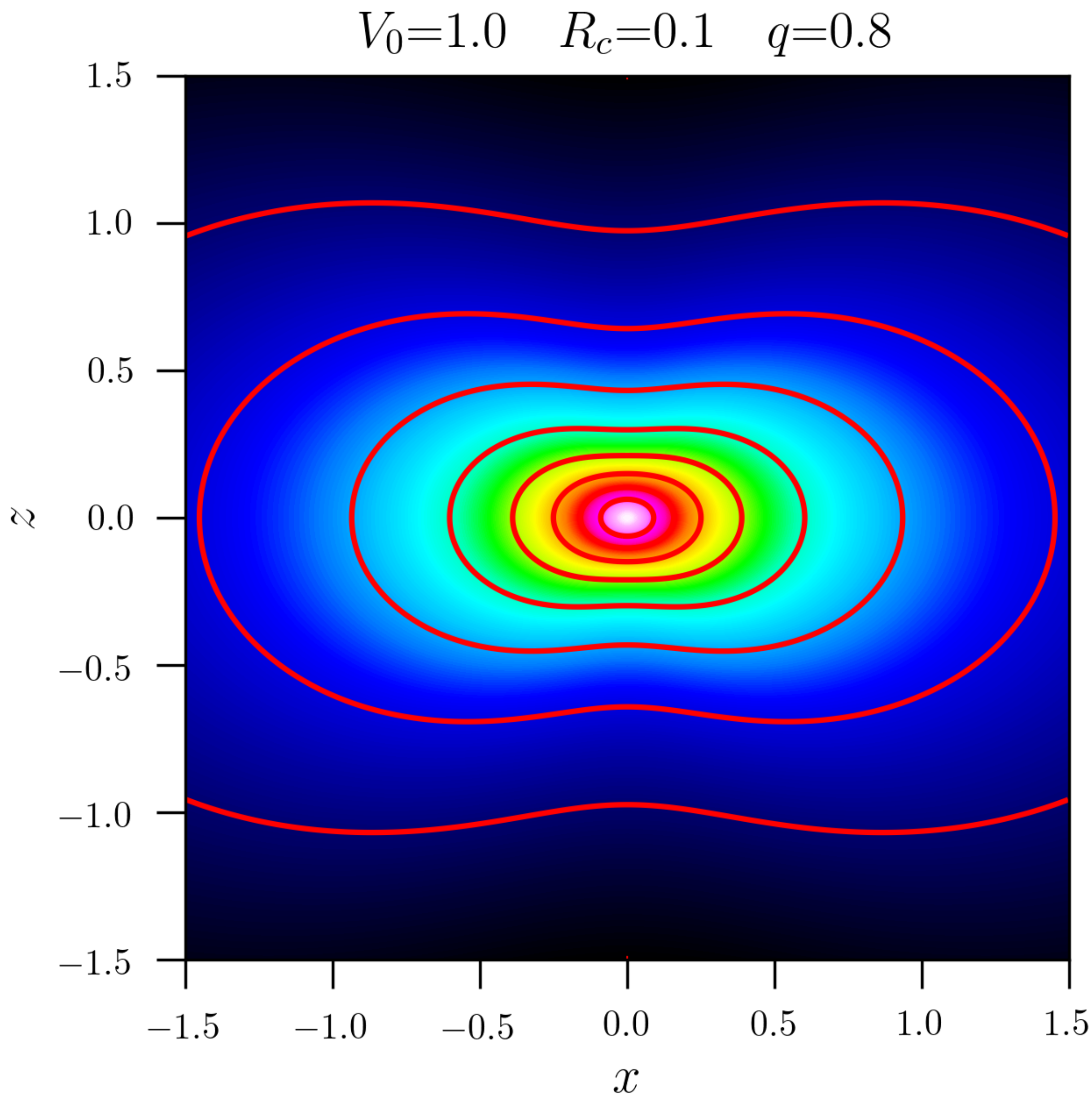
$$V_{c,\log}^2(R) = V_0^2 \frac{R^2}{R_c^2 + R^2}$$

- does not depends on q
- flat rotation curve at large radius

Logarithmic potential

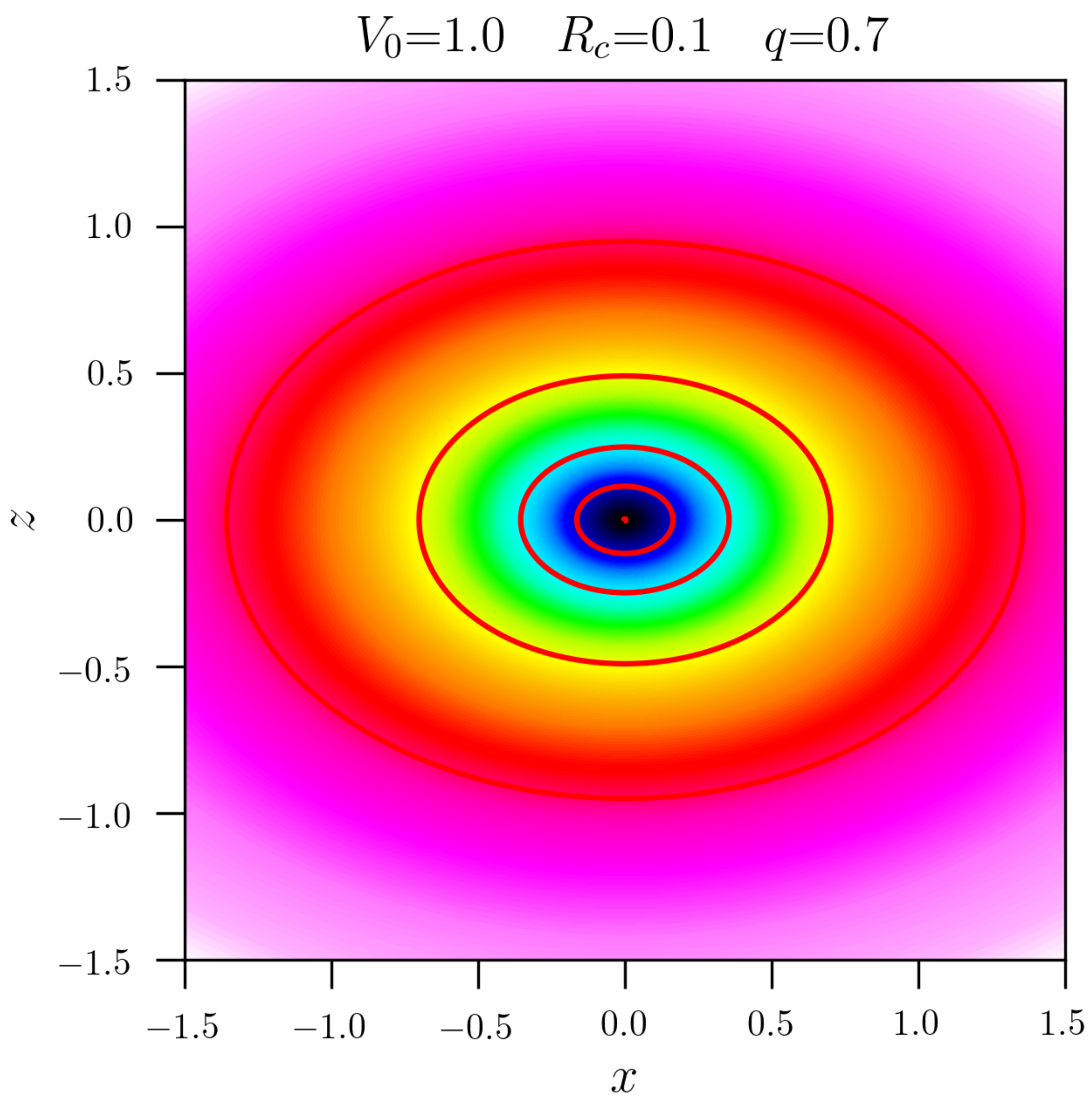


Logarithmic potential

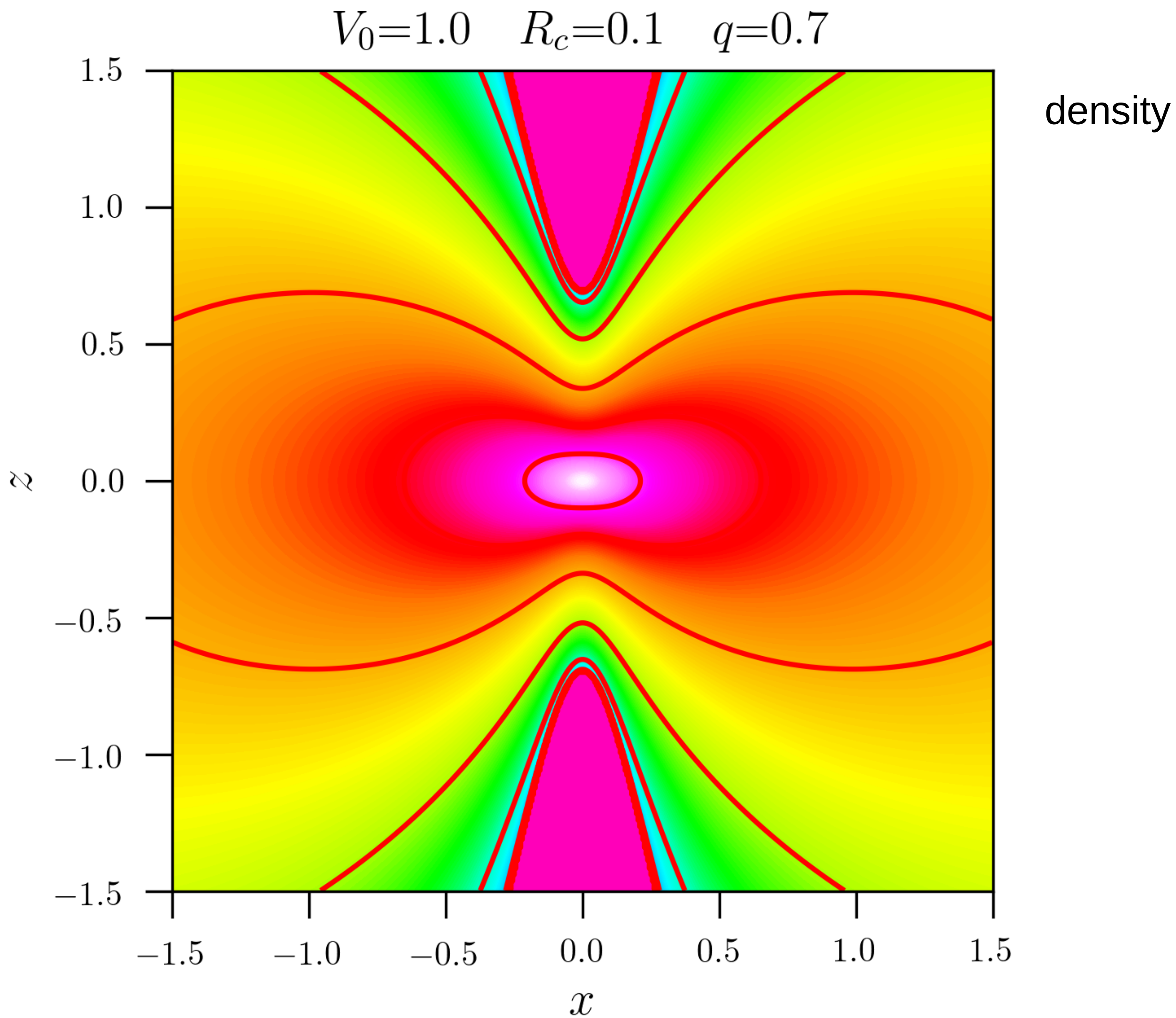


density

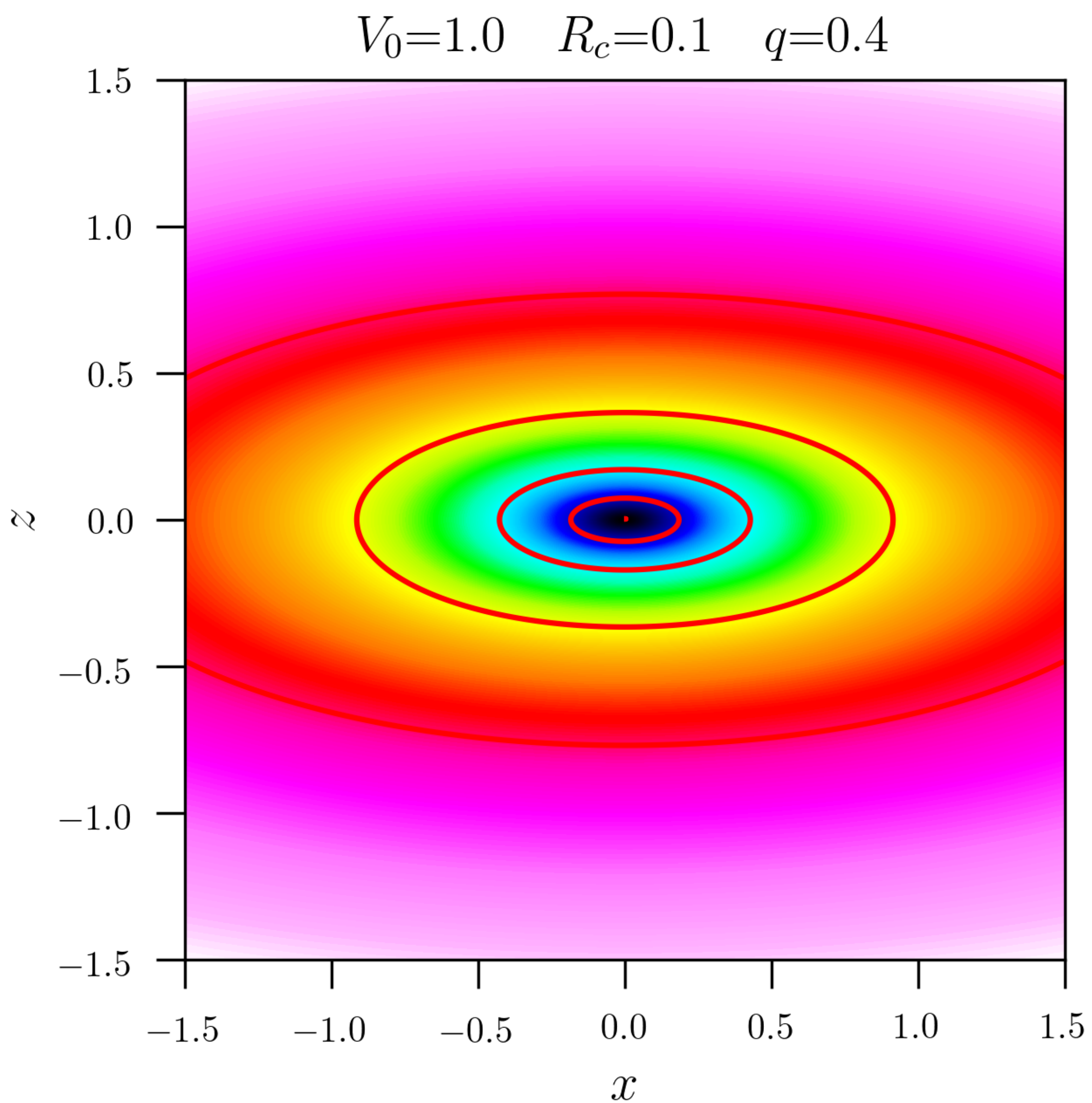
Logarithmic potential



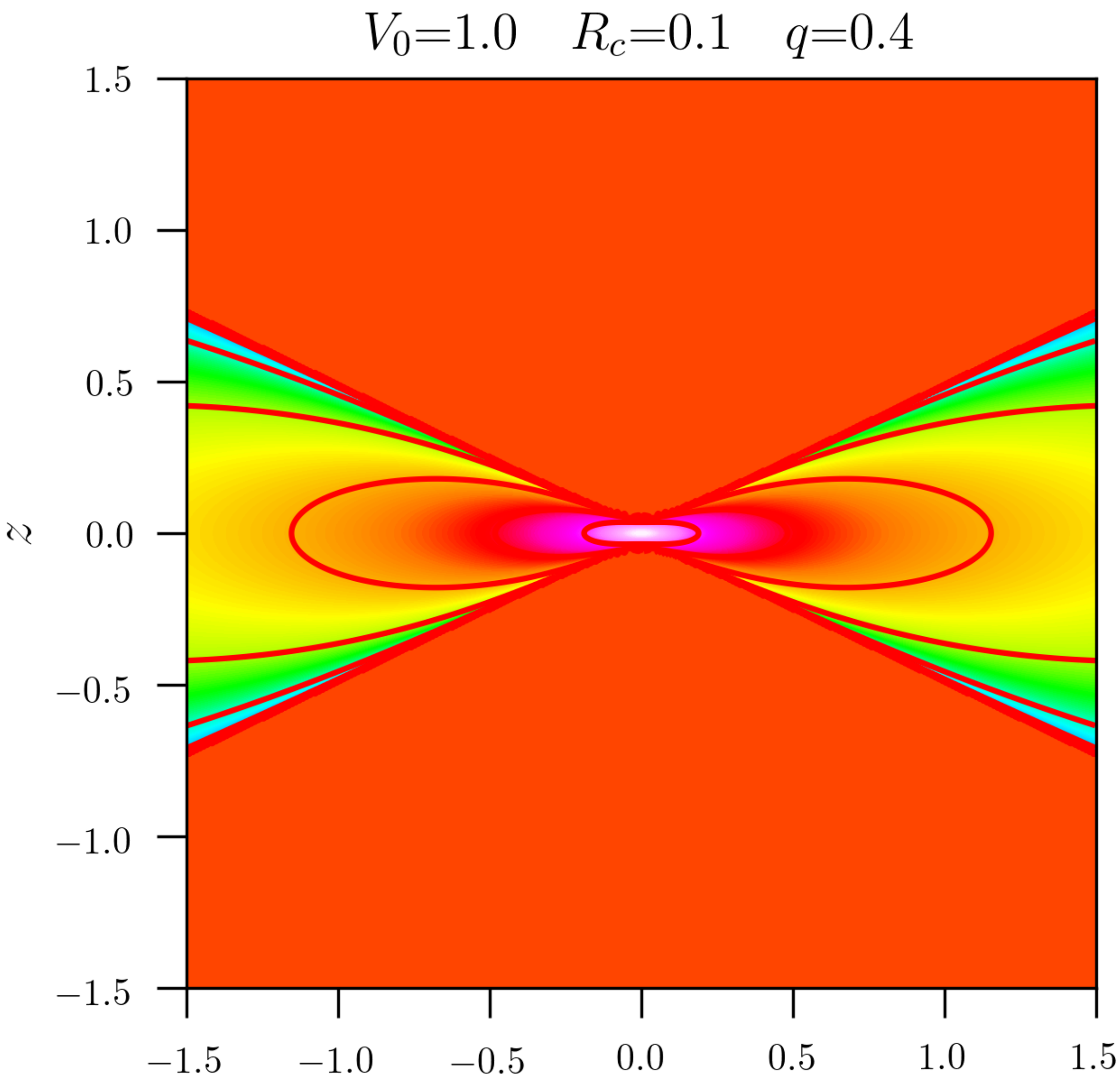
Logarithmic potential



Logarithmic potential



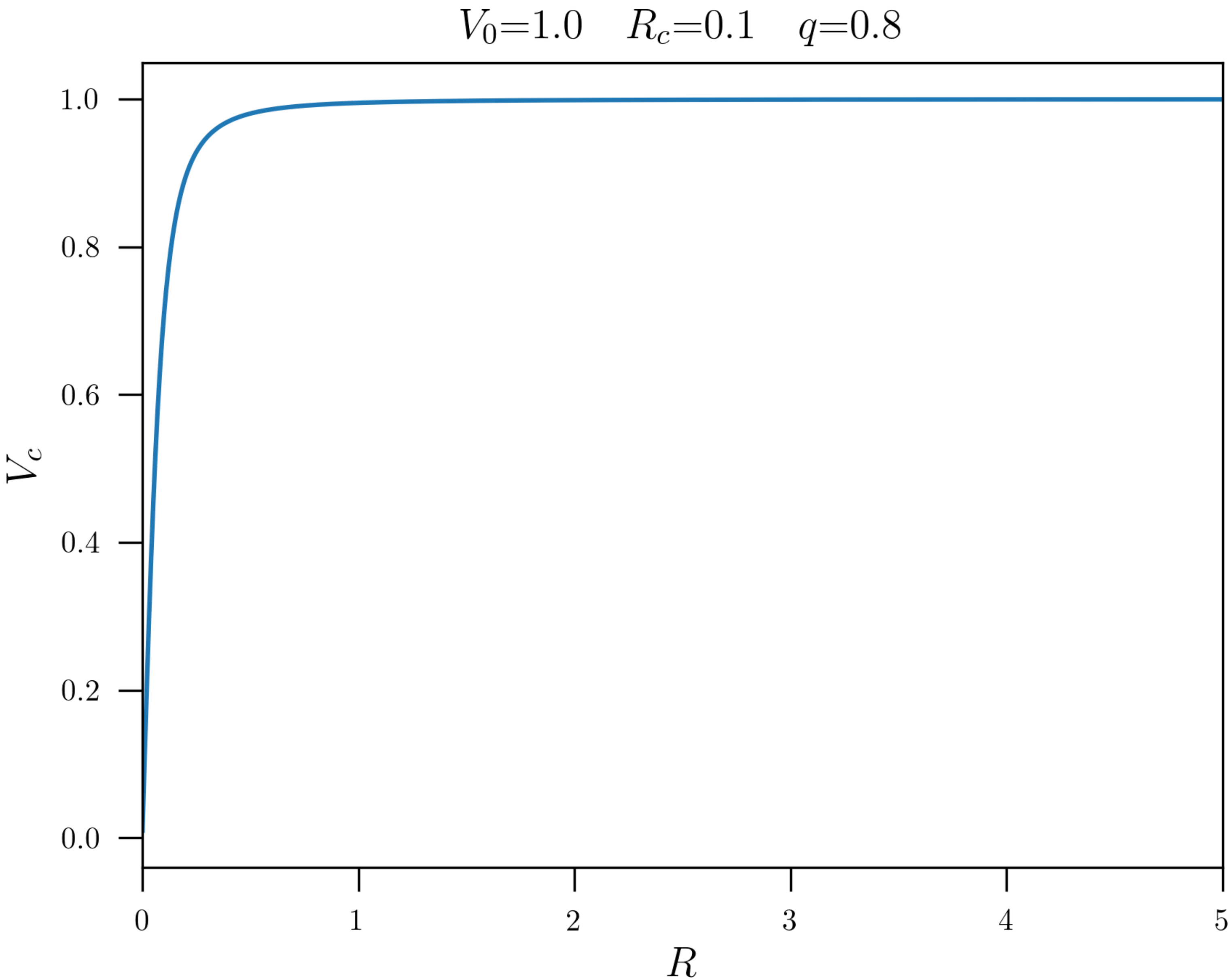
Logarithmic potential



density

Logarithmic potential

Circular velocity rotation curve



The potential of flattened systems

Poisson Equation for very flattened axisymmetric systems

Aim : get $\phi(R, z)$ from $\rho(R, z)$

Poisson equation in cylindrical coord. for axisymmetric systems $\frac{\partial}{\partial \phi}\phi = 0$

$$\nabla^2 \phi(R, z) = \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial \phi}{\partial R}\right) + \frac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho(R, z)$$

What is the behaviour of the Poisson equation when the system get flatter and flatter ?

Example : Miyamoto-Nagai disk **$b \to 0$**

1) $$\rho_{MN}(R, z=0) \overset{b \to 0}{=} \frac{b^2 M}{4\pi} \frac{aR^2 + a^3}{(R^2+a^2)^{5/2}} \frac{1}{b^3} \sim \frac{1}{b} \to \infty$$

2) $$\left.\frac{\partial \phi_{MN}}{\partial R}\right|_{z=0} = \left.\frac{\partial \phi_K}{\partial R}\right|_{z=0} = \frac{GM}{(R^2+a^2)^{3/2}}$$: does not diverge

$$\Rightarrow \frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial \phi}{\partial R}\right)$$: does not diverge

$\Rightarrow$ becomes negligible compared to ρ

Near $z=0$ the Poisson equation becomes

$$\boxed{\frac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho(R, z)}$$

The vertical variation of ϕ depends only on the density ρ at that radius

Solutions of the Poisson equation

$$\phi(R,z) = \underbrace{\phi_0(R,0)}_{\text{"zero point"}} + \underbrace{\phi_z(R,z)}_{\text{"vertical dep."}}$$

1) $$\phi_z(R,z) = 4\pi G \int_0^z dz' \int_0^{z'} dz'' \rho(R,z'')$$

2) $\phi_0(R,0)$ is obtained by assuming a "razor-thin" disk

$$\rho(R,z) \to \Sigma(R)$$

We need a machinery to find $\phi_0(R,0)$ from $\Sigma(R)$

Surface density-based (razor-thin) disks

Kuzmin disk

Kuzmin 1956

$$\Phi_{\rm K}(R,z) = -\frac{GM}{\sqrt{R^2 + (a+|z|)^2}}$$

Plummer based model

$$\Sigma_{\rm K}(R) = \frac{aM}{2\pi(R^2 + a^2)^{3/2}}$$

Infinitely thin disk

$$V^2_{c,{\rm K}}(R) = \frac{G\,M\,R^2}{\left(R^2 + a^2\right)^{3/2}}$$

Equivalent to the Plummer model

$$V^2_{c,{\rm P}}(r) = \frac{GMr^2}{(r^2 + b^2)^{3/2}}$$

Note: for an axi-symmetric model, the circular velocity is computed in the plane z=0.

$$V_c^2(R) = \frac{1}{R}\frac{\rm d\Phi(R, z=0)}{\rm dR}$$

Mestel disk

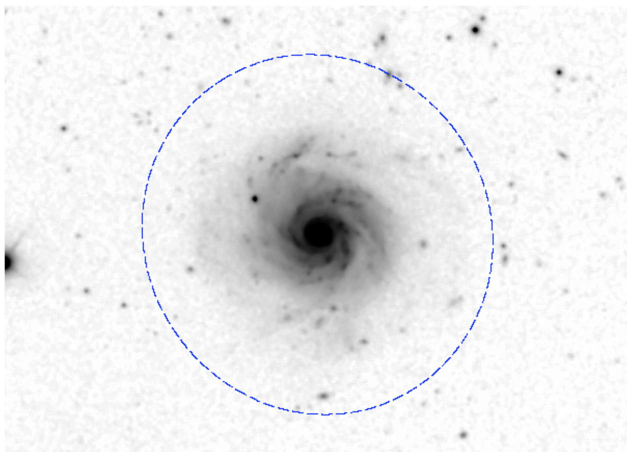
$$\Sigma(R) = \begin{cases} \frac{v_0^2}{2\pi G R} & (R < R_{\rm max}) \\ 0 & (R \geq R_{\rm max}) \end{cases}$$

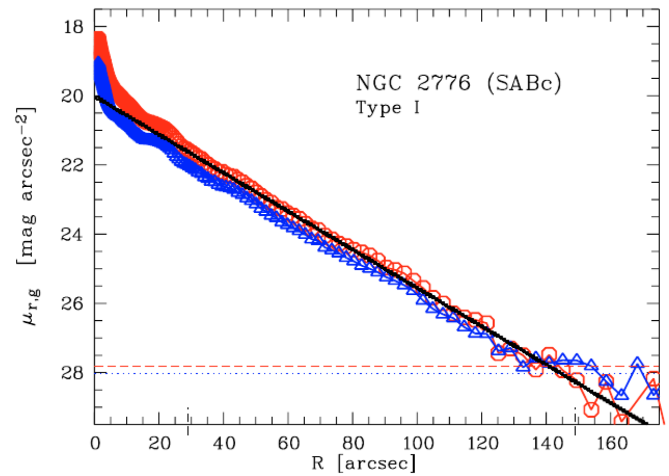
"2D" version of the Isothermal sphere

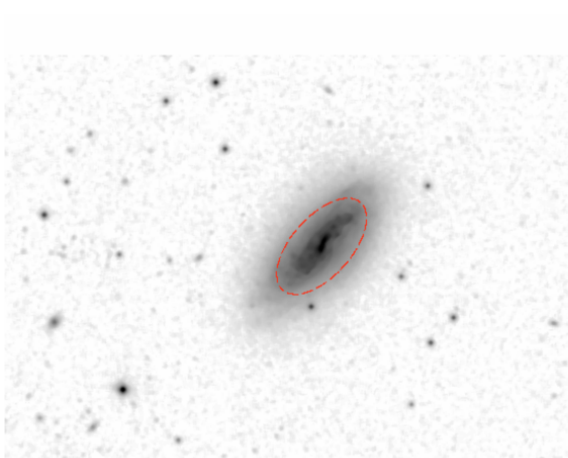
$$\Phi(R, z) = ? \qquad V_{\rm c}(R) = ?$$

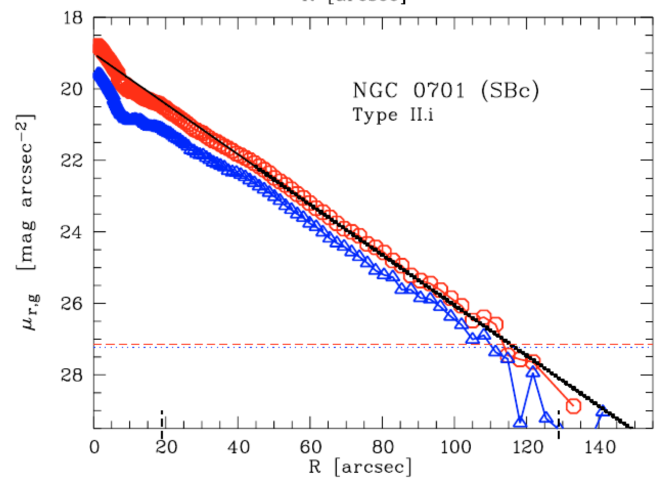
Exponential disk

$$\Sigma(R) = \Sigma_0 \, e^{-R/R_d}$$









$$\Phi(R, z) = ? \qquad V_c(R) = ?$$

Pohlen & Trujillio 2006
See also Freeman 1970

The potential of infinite thin (razor-thin) disks

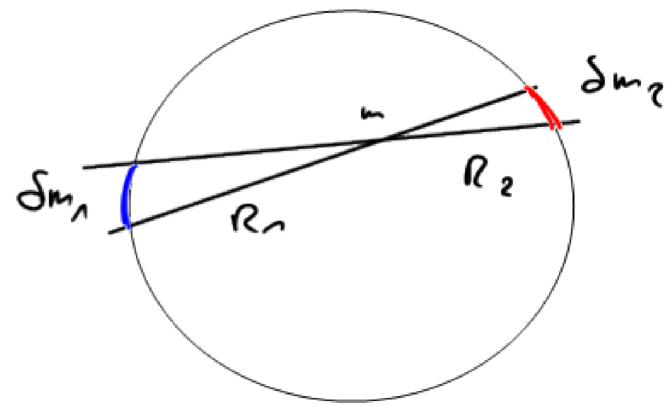
Potential of zero-thickness (razor-thin) disks

Idea: Sum the contribution of a set of rings as we did for spherical models, summing shells

$$\Sigma(R) = \frac{M\,\delta(R-R_0)}{2\pi R_0} \qquad \text{as } M = 2\pi \int_0^\infty \frac{M}{2\pi R_0}\,\delta(R-R_0)\,R\,dR$$

Potential of a ring

no Newton theorem !

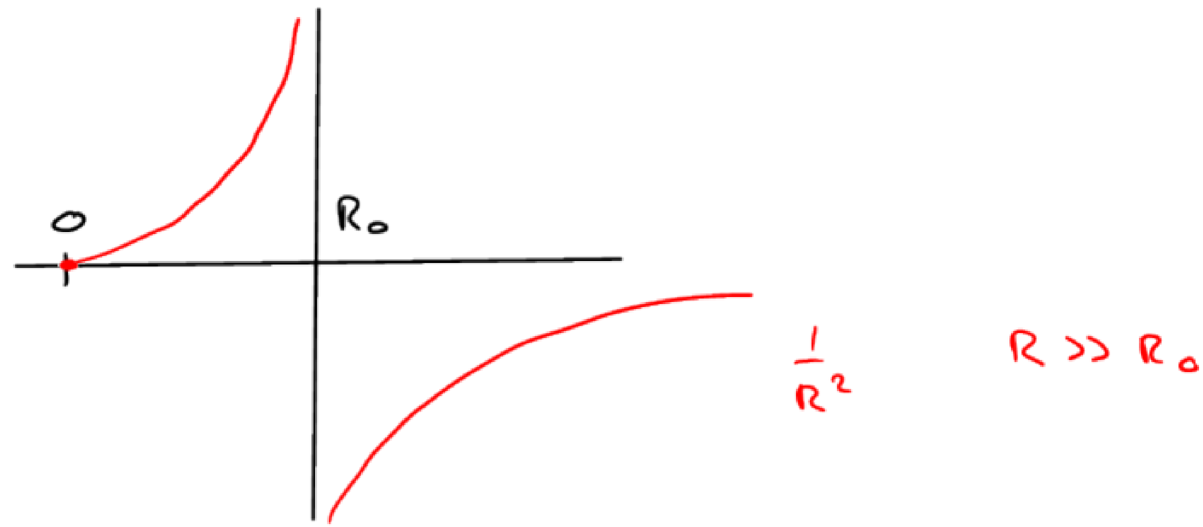


- $\delta m_1 = \Sigma \cdot R_1 \, d\theta \, dR$
- $\delta m_2 = \Sigma \cdot R_2 \, d\theta \, dR$

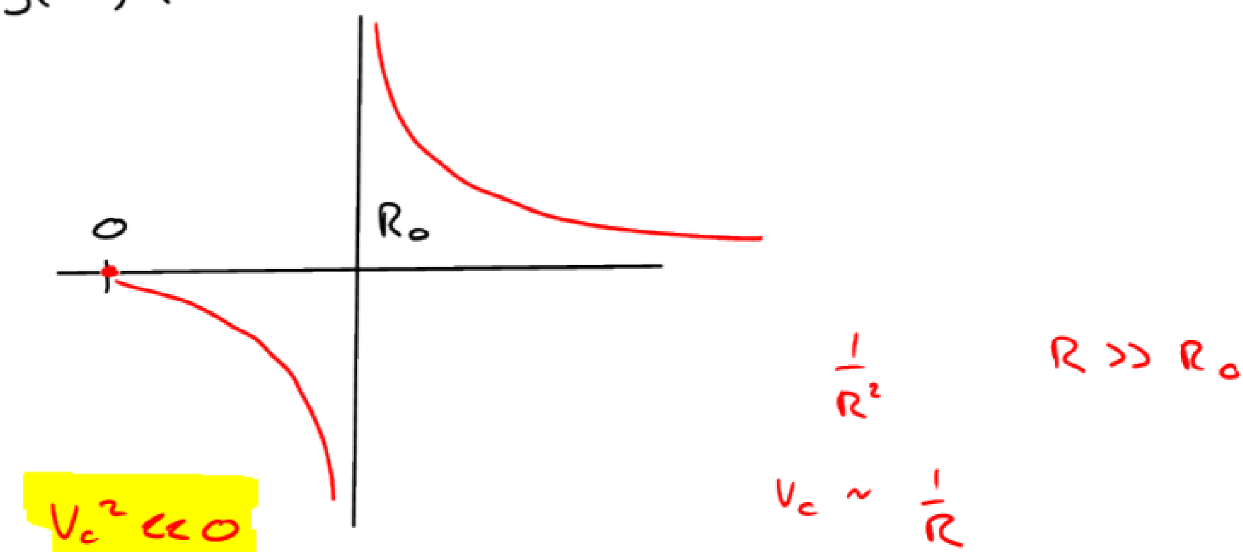
$$\delta F_1 = \frac{G m \Sigma \, d\theta \, dR}{R_1} \neq \frac{G m \Sigma \, d\theta \, dR}{R_1} = \delta F_2$$

Estimation of the gravitational field / circular velocity

$g(R)$



$$V_c^2 = R \frac{\partial \phi}{\partial R} = -g(R)\,R$$



Potential / force of a ring

Lass & Blitzer 1982

$$\phi(R, z) = -\frac{2\,G M K(k)}{\pi\sqrt{(R_0+R)^2 + z^2}} \qquad \left(k^2 = \frac{4 R_0 R}{(R+R_0)^2 + z^2} \right)$$

$$g(R, z) = -\frac{G M}{R\,\pi\sqrt{(R_0+R)^2 + z^2}} \left[k \, \frac{R^2 - R_0^2 - z^2}{4(1-k) R R_0} E(k) + K(k) \right]$$

with • $K(m)$: complete elliptic integral of first kind

$$K(m) = \int_0^{\pi/2} \left[1 - m^2 \sin(t)^2\right]^{-1/2} dt$$

• $E(m)$: complete elliptic integral of second kind

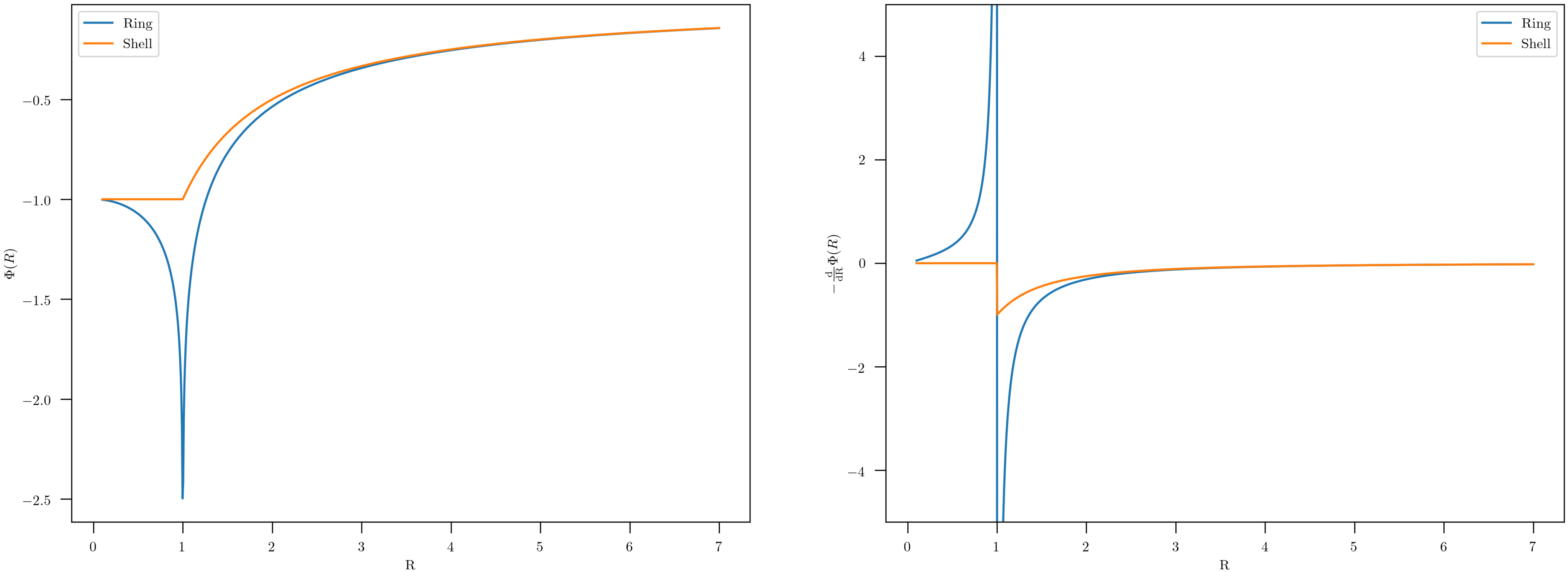
$$E(m) = \int_0^{\pi/2} \left[1 - m^2 \sin(t)^2\right]^{+1/2} dt$$

Potential / force of a ring

Lass & Blitzer 1982

$$\phi(R,z) = -\frac{2GMK(k)}{\pi\sqrt{(R_0+R)^2+z^2}} \qquad \left(k^2 = \frac{4R_0R}{(R+R_0)^2+z^2} \right)$$

$$g(R,z) = -\frac{GM}{R\,\pi\sqrt{(R_0+R)^2+z^2}}\left[k\,\frac{R^2-R_0^2-z^2}{4(1-k)RR_0}\,E(k) + K(k) \right]$$



Potential of a razor-thin disk of surface density $\Sigma(R)$

Sum of rings

$$\phi(R,z) = \int \delta_{R'} \phi(R,z)$$

$$= \int_0^\infty - \frac{2\,G\,\delta M'\,K(k)}{\pi\sqrt{(R'+R)^2 + z^2}} \qquad \text{with } \delta M' = 2\pi\Sigma(R')\,R'\,dR'$$

$$\boxed{\phi(R,z) = -4\,G \int_0^\infty dR' \,\frac{\Sigma(R')\,R'}{\pi\sqrt{(R'+R)^2 + z^2}}\,K(k)}$$

with $k = \sqrt{\dfrac{4R_0R}{(R+R_0)^2+z^2}}$ $\qquad z = 0$

we get

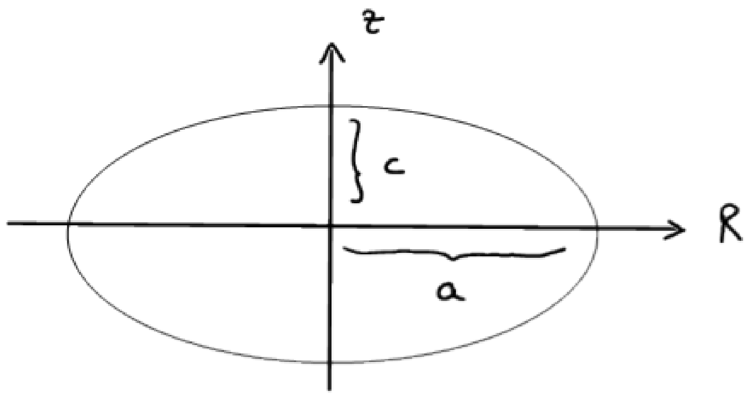
$$\phi(R, z=0) = \frac{-4G}{\sqrt{R}} \int_0^{\infty} dR' \, \Sigma(R') \sqrt{R'} \, k \, K(k)$$

BT 2.265

⚠ This integral has a singularity for $R' \to R$

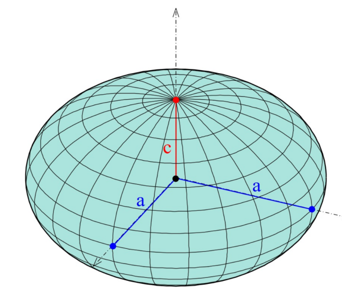
The potential of spheroidal shells (homoeoids)

Spheroids = ellipse of revolution (axisymmetric system)



Equation of an ellipse

$$\frac{R^2}{a^2} + \frac{z^2}{c^2} = 1$$



we assume a constant density ρ

Eccentricity : $e^2 = 1 - \frac{c^2}{a^2}$ $\qquad$ q tq $c = q \cdot a$

Mass $\qquad V = \frac{4}{3}\pi a^2 c \Rightarrow$

$$M(a) = \frac{4}{3}\pi q a^3 \rho$$

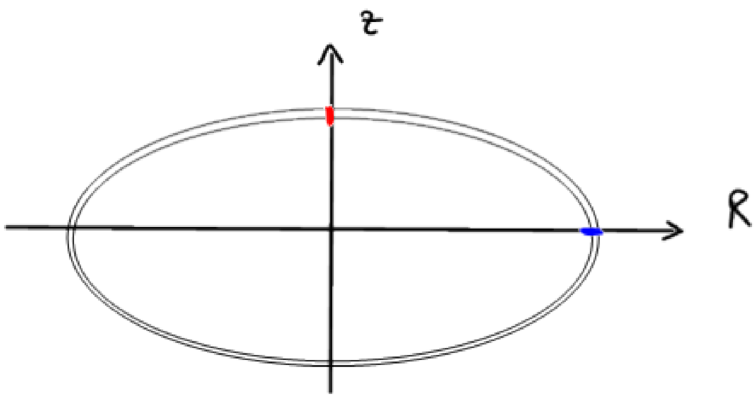
Surface density $\qquad \Sigma(R) = \int_{-\infty}^{\infty} dz\, \rho(R,z) = \rho \int_{-z(R)}^{z(R)} dz = 2\rho\, z(R)$

from the ellipse equation $\quad z^2(R) = c^2\left(1 - \frac{R^2}{a^2}\right) \qquad z(R) = q\sqrt{a^2 - R^2}$

$$\Sigma(R,a) = 2\rho\, q\sqrt{a^2 - R^2}$$

$$\Sigma_0(a) = \Sigma(0,a) = 2\rho q a$$

Homoeoid : Shell of a spheroid of constant density



(i) inner $\frac{R^2}{a^2} + \frac{z^2}{c^2} = 1$

(o) outer $\frac{R^2}{a^2} + \frac{z^2}{c^2} = (1 + \delta\beta)^2$

For $z = 0$

(i) $R = a$

(o) $R = a + a\,\delta\beta$

$\Delta R = a\,\delta\beta$

For $R = 0$

(i) $z = c$

(o) $z = c + c\,\delta\beta$

$\Delta z = c\,\delta\beta$

$\Rightarrow$ the thickness of the shell varies

Mass

$M(a)$ = mass of a spheroid

$$\delta M(a) = \frac{dM}{da}\delta a = 4\pi q \rho a^2 \delta a = 2\pi a \Sigma_0(a)\,\delta a$$

$$\boxed{\delta M(a) = 2\pi a \Sigma_0(a)\,\delta a}$$

Surface density

$\Sigma(a)$ = surface density of a spheroid

$$\delta \Sigma(a) = \frac{d\Sigma}{da}\delta a = \frac{2\rho q a}{-\sqrt{a^2 - R^2}}\delta a = \frac{\Sigma_0(a)}{-\sqrt{a^2-R^2}}\delta a$$

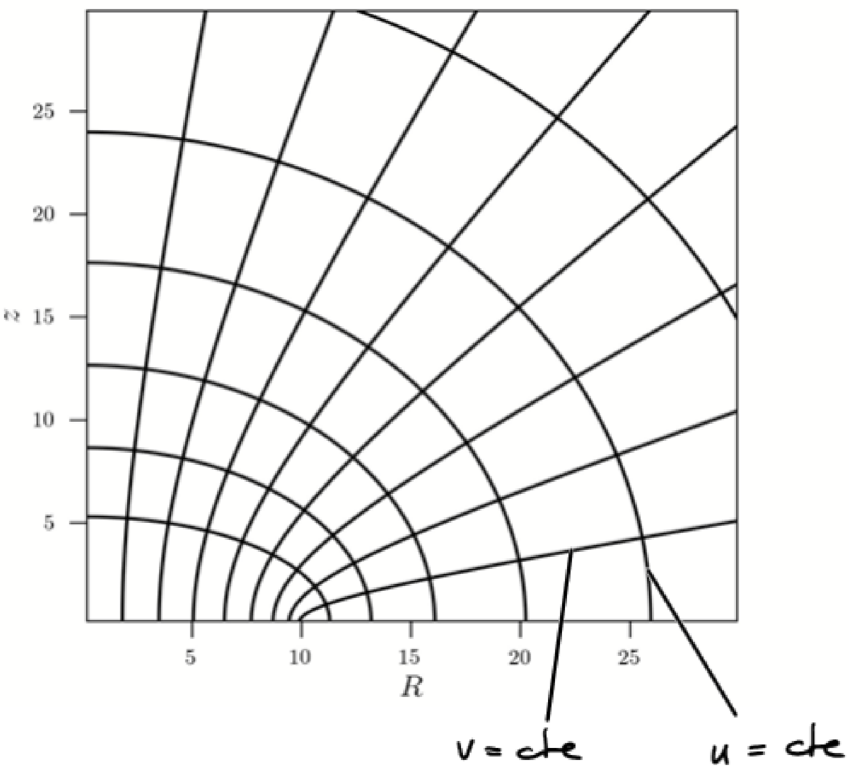
$$\boxed{\delta \Sigma(a) = \frac{\Sigma_0(a)}{-\sqrt{a^2-R^2}}\delta a}$$

Potential of homoeoids

Spheroidal coordinates

positive constant ↓

$$(R, \phi, z) \to (u, \phi, v) \qquad \begin{cases} R = \Delta \cosh u \, \sin v \\ z = \Delta \sinh u \, \cos v \end{cases}$$



For a constant u (removing v)

$$\frac{R^2}{\Delta^2 \cosh^2 u} + \frac{z^2}{\Delta^2 \sinh^2 u} = 1$$

Eccentricity $\quad e^2 = 1 - \frac{\sinh^2 u}{\cosh^2 u} = 1 - \tanh^2 u$

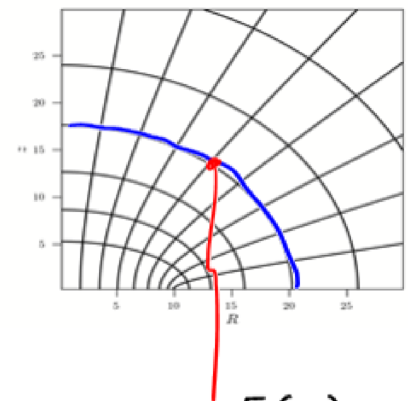
$$\begin{cases} u = 0 & e = 1 \\ u = \infty & e = 0 \end{cases}$$

$$\begin{cases} a = \Delta \cosh u \\ c = \Delta \sinh u \end{cases}$$

It is possible to demonstrate that

1) the surface density of an thin homoeoid of "radius" u_0 and mass δM

$a = \Delta \cosh u$



$$\Sigma(v) = \frac{\delta M}{4\pi a^2 \sqrt{1 - e^2 \sin^2 v}}$$

2) its corresponding potential is

$$\phi(u) = -\frac{G\delta M}{ae} \begin{cases} \arcsin(e) & u \leq u_0 \\ \arcsin\left(\frac{1}{\cosh(u)}\right) & u \geq u_0 \end{cases}$$

Potential of an homoeoid

Assume $\phi = \phi(u)$ and try to solve $\nabla^2 \phi = 0$

for $\phi = \phi(u)$

$$\nabla^2 \phi = \frac{1}{\Delta^2 (\sinh^2 u + \cos^2 v)} \left[\frac{1}{\cosh u} \frac{\partial}{\partial u} \left(\cosh u \frac{\partial \phi}{\partial u} \right) \right] = 0$$

$$\frac{\partial}{\partial u} \left(\cosh u \frac{\partial \phi}{\partial u} \right) = 0$$

$$\left(\begin{array}{l} \arcsin' \to \frac{1}{\sqrt{1-u^2}} \\ \cosh' \to \sinh u \end{array} \right)$$

solutions

1) $\phi = \phi_0 = \text{cte}$

2) $\phi = -A \arcsin\left(\frac{1}{\cosh u} \right) + B$

for $u \to \infty$, using $R = \Delta \cosh u \cdot \sin v$

$$z = \Delta \sinh u \cos v$$

and $\cosh^2 u = \sinh^2 u \quad (u \to \infty)$

we get $r^2 = R^2 + z^2 = \Delta^2 (\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v)$

$$= \Delta^2 \cosh^2 u$$

$$\Rightarrow \quad \frac{1}{\cosh(u)} \to \frac{\Delta}{r}$$

So, $-A \arcsin\left(\frac{1}{\cosh(u)}\right) + B \simeq -A \arcsin\left(\frac{\Delta}{r}\right) + B \simeq -\frac{\Delta}{r} + B$

$$\Rightarrow \quad A = \frac{G\delta M}{\Delta} \qquad B = 0$$

So, we get the potential:

$$\phi(u) = -\frac{G\delta M}{\Delta} \begin{cases} \arcsin\left(\frac{1}{\cosh(u_0)}\right) & u < u_0 \\ \arcsin\left(\frac{1}{\cosh(u)}\right) & u > u_0 \end{cases}$$

u_0 is the surface of an ellipsoid of semi-major/minor axis

$$\begin{cases} a = \Delta \cosh u_0 \\ c = \Delta \sinh u_0 \end{cases} \Rightarrow e = \sqrt{1 - \frac{\cosh^2 u_0}{\sinh^2 u_0}} = \frac{1}{\cosh u_0}$$

and $ae = \Delta$

$$\phi(u) = -\frac{G\delta M}{ae} \begin{cases} \arcsin(e) & u < u_0 \\ \arcsin\left(\frac{1}{\cosh(u)}\right) & u > u_0 \end{cases}$$

COMPLEMENT

What is the density at the surface u_0 ?

Gauss theorem :

$$\int \vec{\nabla}\phi \, d^2\vec{s} = 4\pi G M$$

$$\delta M = \frac{\vec{\nabla}\phi \cdot \vec{e}_n \, ds^2}{4\pi G}$$

$$\Sigma(u) = \frac{\delta M}{ds^2} = \frac{\vec{\nabla}\phi \cdot \vec{e}_n}{4\pi G}$$

In elliptical coord :

$$\vec{\nabla}\phi(u) = \frac{1}{\Delta\sqrt{\sinh^2 u + \cos^2 v}} \frac{\partial \phi}{\partial u} \vec{e}_n$$

so

$$\Sigma(u) = \frac{1}{\Delta\sqrt{\sinh^2 u + \cos^2 v}} \left.\frac{\partial \phi}{\partial u}\right|_{u=u_0} \frac{1}{4\pi G}$$

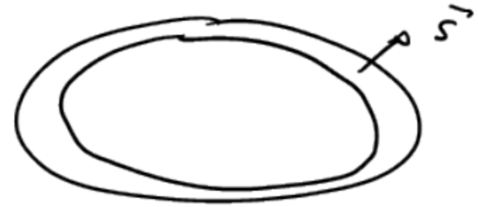
$$\boxed{\Sigma(u) = \frac{\delta M}{4\pi a^2 \sqrt{1 - e^2 \sin^2 v}}}$$

COMPLEMENT

Link between $\Sigma(u)$ and the sof. density of an homoeoid

$$\beta^2 = \beta^2(R,z) = \frac{R^2}{a^2} + \frac{z^2}{c^2}$$

$$\vec{s} = s \cdot \vec{e}_n = s \cdot \frac{\vec{\nabla}\beta}{|\vec{\nabla}\beta|}$$



$|\vec{s}| = s =$ thickness of the shell

$$\delta\beta = \vec{s} \cdot \vec{\nabla}\beta \qquad \beta(R,z) = \beta(R_0,z_0) + \vec{\nabla}\beta \cdot \vec{s}$$

$$\tfrac{1}{2}(\ )^{-\frac{1}{2}}\, \frac{2R}{a^2}$$

$$= s\,|\vec{\nabla}\beta|$$

$$s = \frac{\delta\beta}{|\vec{\nabla}\beta|} \longrightarrow \begin{cases} \vec{\nabla}\beta = \dfrac{\partial\beta}{\partial R}\vec{e}_R + \dfrac{\partial\beta}{\partial z}\vec{e}_z = \dfrac{1}{\beta}\dfrac{R}{a^2}\,\vec{e}_R + \dfrac{1}{\beta}\dfrac{z}{c^2}\,\vec{e}_z \\ |\nabla\beta| = \sqrt{\dfrac{R^2}{a^4} + \dfrac{z^2}{c^4}}\ \beta^{-1} \end{cases}$$

$$s = \left(\frac{R^2}{a^4} + \frac{z^2}{c^4}\right)^{-\frac{1}{2}} \beta\, \delta\beta$$

$$\Sigma = \rho \cdot s = \rho \left(\frac{R^2}{a^4} + \frac{z^2}{c^4}\right)^{-\frac{1}{2}} \beta\, \delta\beta$$

we introduce v, such that : $R = \beta a \sin(v)$ $\quad z = \beta c \cos(v)$

$$e = \sqrt{1 - \frac{c^2}{a^2}} \qquad e^2 = 1 - \frac{c^2}{a^2}$$

$$e^2 a^2 = a^2 - c^2$$

$$\left(\frac{R^2}{a^4} + \frac{z^2}{c^4}\right)^{-\frac{1}{2}} \beta \, \rho \, d\beta$$

$$R^2 = \beta^2 a^2 \sin^2 v$$

$$z^2 = \beta^2 c^2 \cos^2 v$$

$$\frac{R^2}{a^4} + \frac{z^2}{c^4}$$

$$\beta^2 \frac{1}{a^2} \sin^2 v + \beta^2 \frac{1}{c^2} \cos^2 v$$

$$\beta^2 \left(\frac{\sin^2 v}{a^2} + \frac{\cos^2 v}{c^2} \right)$$

$$c^2 = a^2(1 - e^2)$$

$$\beta^2 \left(\frac{\sin^2 v}{a^2} + \frac{1}{c^2} - \frac{\sin^2 v}{c^2} \right)$$

$$c = a\sqrt{1 - e^2}$$

$$\frac{\beta^2}{a^2 c^2} \left(c^2 \sin^2 v + a^2 - a^2 \sin^2 v \right)$$

COMPLEMENT

$$\frac{\rho^2}{a^2c^2}\left(\sin^2 v\,(c^2-a^2) + a^2\right)$$

$$\frac{\rho^2}{a^{2\cancel{4}}(1-e^2)}\left(\cancel{a^2} - e^2\cancel{a^2}\sin^2 v\right)$$

$$\Sigma = \rho\left(\frac{R^2}{a^4}+\frac{t^2}{c^4}\right)^{-\frac{1}{2}}\rho\,\delta\rho$$

$$= \frac{a\sqrt{1-e^2}}{\cancel{\rho}\sqrt{1-e^2\sin^2 v}}\,\rho\,\cancel{\rho}\,\delta\rho = \frac{a\sqrt{1-e^2}\,\rho\,\delta\rho}{\sqrt{1-e^2\sin^2 v}}$$

Volume of the ellipsoid : $V = \frac{4}{3}\pi a^2 c\rho^3 = \frac{4}{3}\pi a^3\rho^3\sqrt{1-e^2}$

$$\delta M = \delta(\rho M) = 4\pi a^3\rho^2\rho\sqrt{1-e^2}\,\delta\rho$$

introduce $\delta\rho$ in $\Sigma = \dfrac{a\sqrt{1-e^2}\,\rho\,\delta\rho}{\sqrt{1-e^2\sin^2 v}}$

and set $\rho = 1$

COMPLEMENT

$$\Sigma(\nu) = \frac{a\sqrt{1-e^2}\,\rho\,\delta M}{4\pi a^3 \beta^2 \rho \sqrt{1-e^2}\,\sqrt{1-e^2\sin^2\nu}}$$

$$\boxed{\Sigma(\nu) = \frac{\delta M}{4\pi a^2 \sqrt{1-e^2\sin^2\nu}}}$$

#

COMPLEMENT

Newton’s Theorems

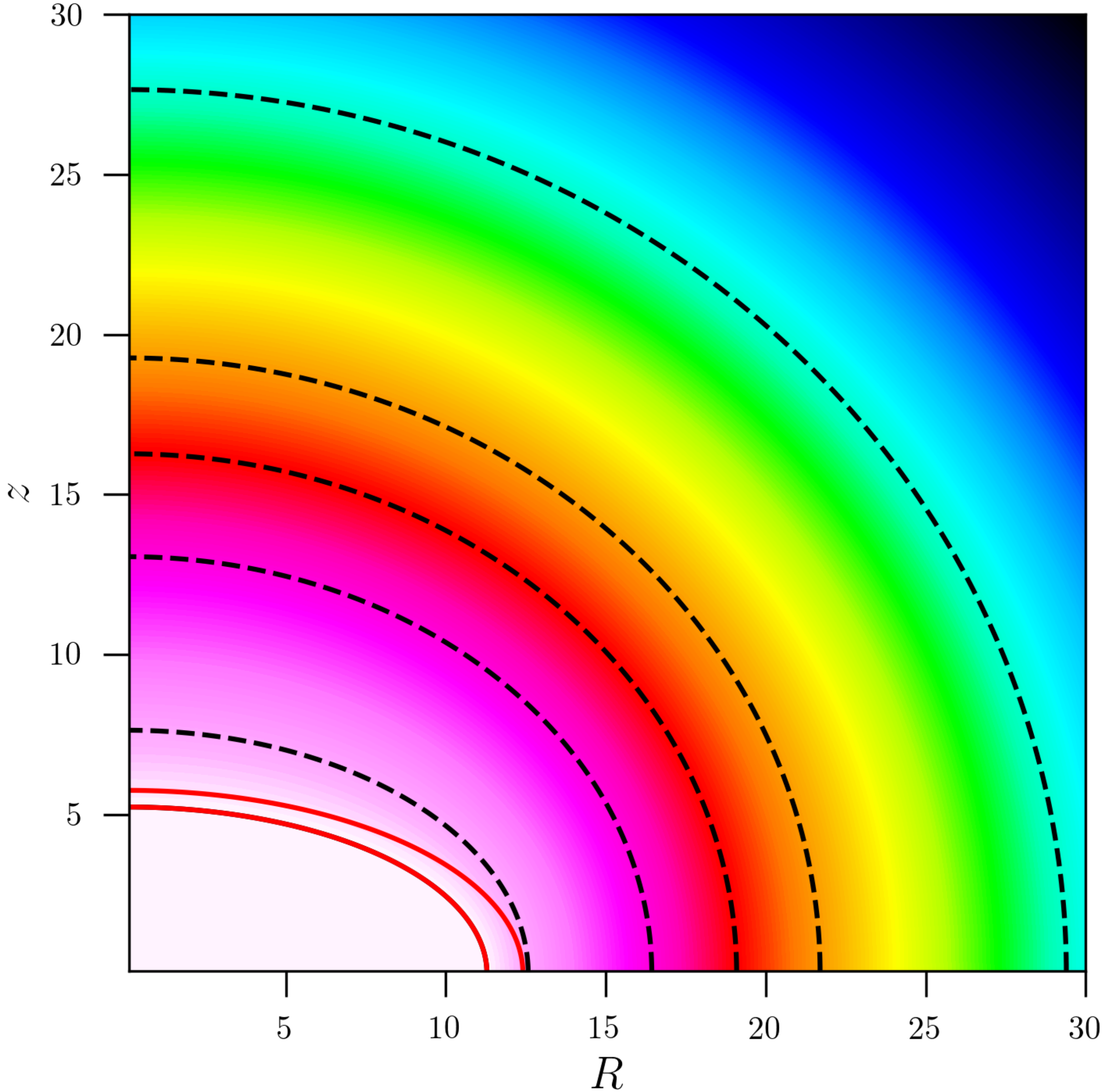
Homoeoid theorem:

- The exterior iso-potential surfaces of a thin homoeoid are the spheroids that are confocal (u=constant) with the shell itself. Inside the shell, the potential is constant.

Newton’s third theorem:

- A mass that is inside a homoeoid experiences no net gravitational force from the homoeoid.

potential of homoeoids



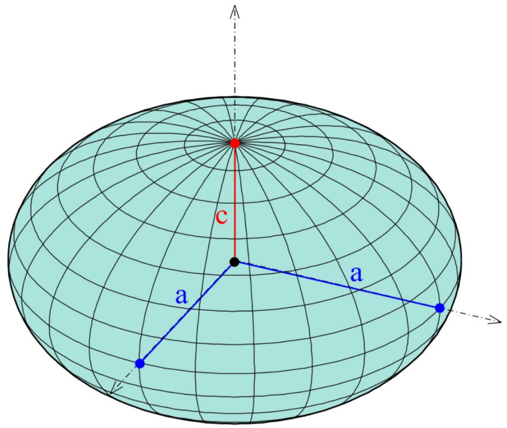
The potential of spheroids

COMPLEMENT

The potential of spheroids defined by

$$\text{constant} = m^2 \equiv R^2 + \frac{z^2}{1-e^2}$$

of density $\rho(m^2)$
may be obtained by summing homoeoids



$$\Phi(R_0, z_0) = -2\pi G \frac{\sqrt{1-e^2}}{e}$$
$$\times \left(\psi(\infty) \sin^{-1} e - \frac{a_0 e}{2} \int_0^\infty \mathrm{d}\tau \, \frac{\psi(m)}{(\tau + a_0^2)\sqrt{\tau + c_0^2}} \right)$$

with:

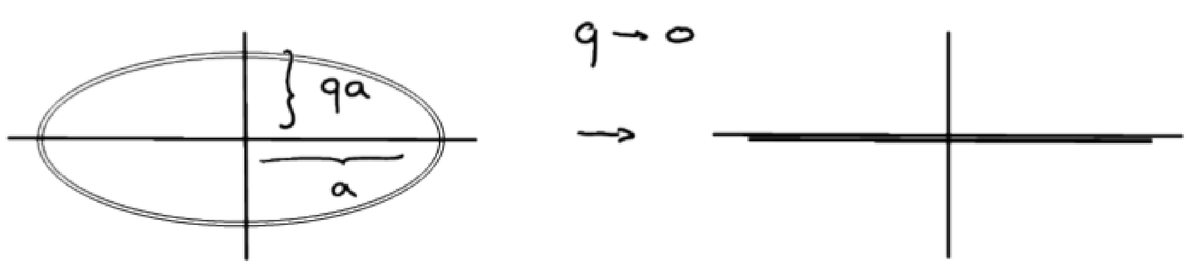
$$\psi(m) \equiv \int_0^{m^2} \mathrm{d}m^2 \, \rho(m^2)$$

The potential of infinite thin (razor-thin) disks from homoeoids

The potential of zero thickness (razor-thin) disks from homoeoids

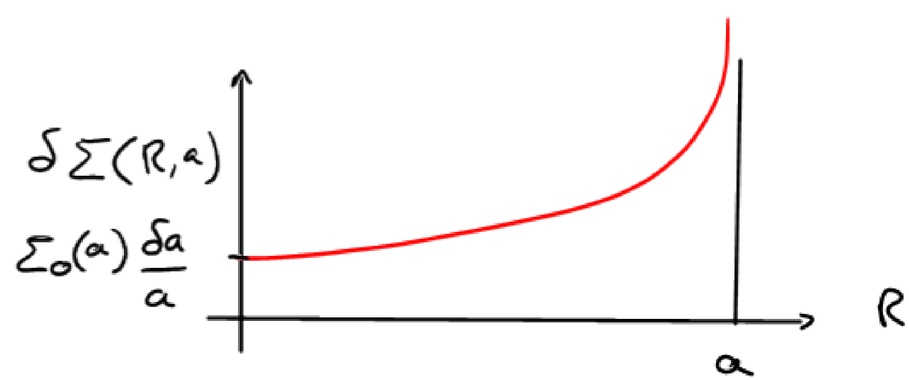
Idea Reproduce any surface density $\Sigma(R)$ by summing a set of infinitely flattened homoeoids

Infinitely flattened homoeoids



The surface density remains the same (indep. of q)

$$\delta\Sigma(a,R) = \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}}\,\delta a$$



Summing infinitely flattened homoeoids

$$\Sigma(R) = \sum_{a \geq R} \delta\Sigma(a,R) = \sum_{a \geq R} \frac{d}{da}\Sigma(a,R)\,\delta a = \sum_{a \geq R} \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}}\,\delta a$$

$$= \int_R^\infty \frac{\Sigma_0(a)}{\sqrt{a^2 - R^2}}\,da$$

Abel integral

Solution:

$$\Sigma_0(a) = -\frac{2}{\pi}\frac{d}{da}\left(\int_a^\infty dR \,\frac{R\,\Sigma(R)}{\sqrt{R^2 - a^2}}\right)$$

For a given $\Sigma(R)$ we can compute $\Sigma_0(a)$ (the weights)

such that $\Sigma(R) = \int_R^\infty \delta\Sigma(a,R)$

Potential of infinitely flattened homoeoids

$$\phi(u) = -\frac{G\delta M}{ae} \arcsin\left(\frac{1}{\cosh(u)}\right) \qquad u \geq u_0$$

The potential is continuous across the plane $z=0$
we can compute it just above the plane
i.e, outside the shell

with $\delta M = 2\pi a \Sigma_0(a)\, \delta a$ and for $u \geq u_0$

and nolting that for $q \to 0$ $e \to 1$

$$\delta\phi_a(R,z) = -\frac{G\, 2\pi a\, \Sigma_0(a)\, \delta a}{ae} \arcsin\left(\frac{1}{\cosh(u)}\right)$$

$$= -2\pi G \Sigma_0(a)\, \delta a \arcsin\left(\frac{1}{\cosh(u)}\right)$$

Expression for u from $\begin{cases} R = \Delta \cosh u \sin v \\ z = \Delta \sinh u \cos v \end{cases}$ and $\cos^2 v + \sin^2 v = 1$

$$\cosh^2 u = \frac{1}{4a^2}\left[\underbrace{\sqrt{z^2+(a+R)^2}}_{\sqrt{+}} + \underbrace{\sqrt{z^2\ (a-R)^2}}_{\sqrt{-}}\right]^2$$

Thus

$$\delta\phi_a(R,z) = -2\pi G\,\Sigma_0(a)\,\arcsin\left(\frac{2a}{\sqrt{+}+\sqrt{-}}\right)\delta a$$

Summing the contribution of all homoeoids

$$\phi(R,z) = \int_0^\infty \delta\phi_a(R,z) = -2\pi G\int_0^\infty \Sigma_0(a)\,\arcsin\left(\frac{2a}{\sqrt{+}+\sqrt{-}}\right)da$$

but $\Sigma_0(a) = -\frac{2}{\pi}\frac{d}{da}\left(\int_a^\infty dR'\,\frac{R'\Sigma(R')}{\sqrt{R'^2-a^2}}\right)$

$$\phi(R,z) = 4G\int_0^\infty da\,\arcsin\left(\frac{2a}{\sqrt{+}+\sqrt{-}}\right)\frac{d}{da}\left(\int_a^\infty dR'\,\frac{R'\Sigma(R')}{\sqrt{R'^2-a^2}}\right)$$

dep. only on "a" : can be tabulated

Integrating by part gives

$$\phi(R,z) = -2\sqrt{2}\,G \int_0^{\infty} da \, \frac{\left[(a+R)/\sqrt{+}\right] - \left[(a-R)/\sqrt{-}\right]}{\sqrt{R^2 - z^2 - a^2 + \sqrt{+}\cdot\sqrt{-}}} \int_a^{\infty} dR' \, \frac{R'\Sigma(R')}{\sqrt{R'^2 - a^2}}$$

Circular velocity

$$V_c^2(R) = R \left.\frac{\partial \phi}{\partial R}\right|_{z=0}$$

$$\frac{d}{dR} \arcsin\left(\frac{2a}{|a+R| + |a-R|}\right)$$

- $R < a \rightarrow a+R+a-R = 2a \Rightarrow \arcsin(1) \rightarrow \frac{d}{dR} = 0$
- $R > a \rightarrow a+R-a+R = 2R \Rightarrow \arcsin\left(\frac{a}{R}\right) \Rightarrow \frac{d}{dR} = -\frac{a/R^2}{\sqrt{1-\left(\frac{a}{R}\right)^2}}$

$$V_c^2(R) = -4G \int_0^{R} da \, \frac{a}{\sqrt{R^2 - a^2}} \, \frac{d}{da}\left(\int_a^{\infty} dR' \, \frac{R'\Sigma(R')}{\sqrt{R'^2 - a^2}} \right)$$

Exponential disk

$$\Sigma(R) = \Sigma_0 \, e^{-R/R_d}$$

The integral in the razor-thin potential equation is then:

$$\int_a^\infty R' \frac{R'\Sigma_0 e^{-R'/R_d}}{\sqrt{R'^2 - a^2}} = \Sigma_0 a K_1(a/R_d)$$

The potential:

$$\Phi(R,z) = -2\sqrt{2}\, G \int_0^\infty a \frac{\frac{a+R}{\sqrt{z^2+(a+R)^2}} - \frac{a-R}{\sqrt{z^2+(a-R)^2}}}{\sqrt{R^2 - z^2 - a^2 + \sqrt{z^2+(a+R)^2}\sqrt{z^2+(a-R)^2}}} \times \Sigma_0 a K_1(a/R_d)$$

The circular velocity:

$$v_c^2 = 4\pi G \Sigma_0 R_d y^2 \left[I_0(y)K_0(y) - I_1(y)K_1(y)\right]$$

$$y = \frac{R}{2\,R_\mathrm{d}}$$

Bessel functions

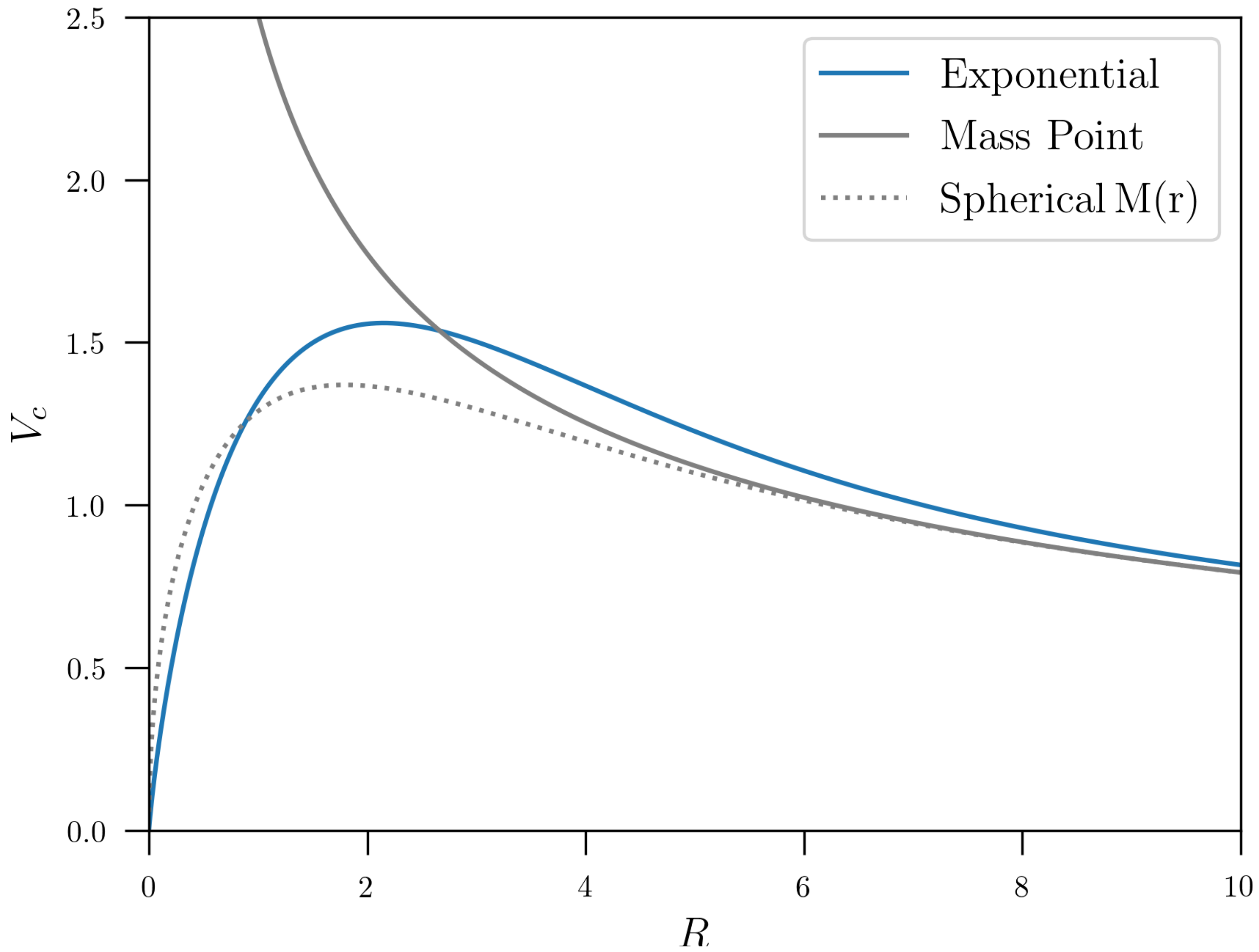
$$I_\nu(z) = i^{-\nu} J_\nu(iz)$$

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

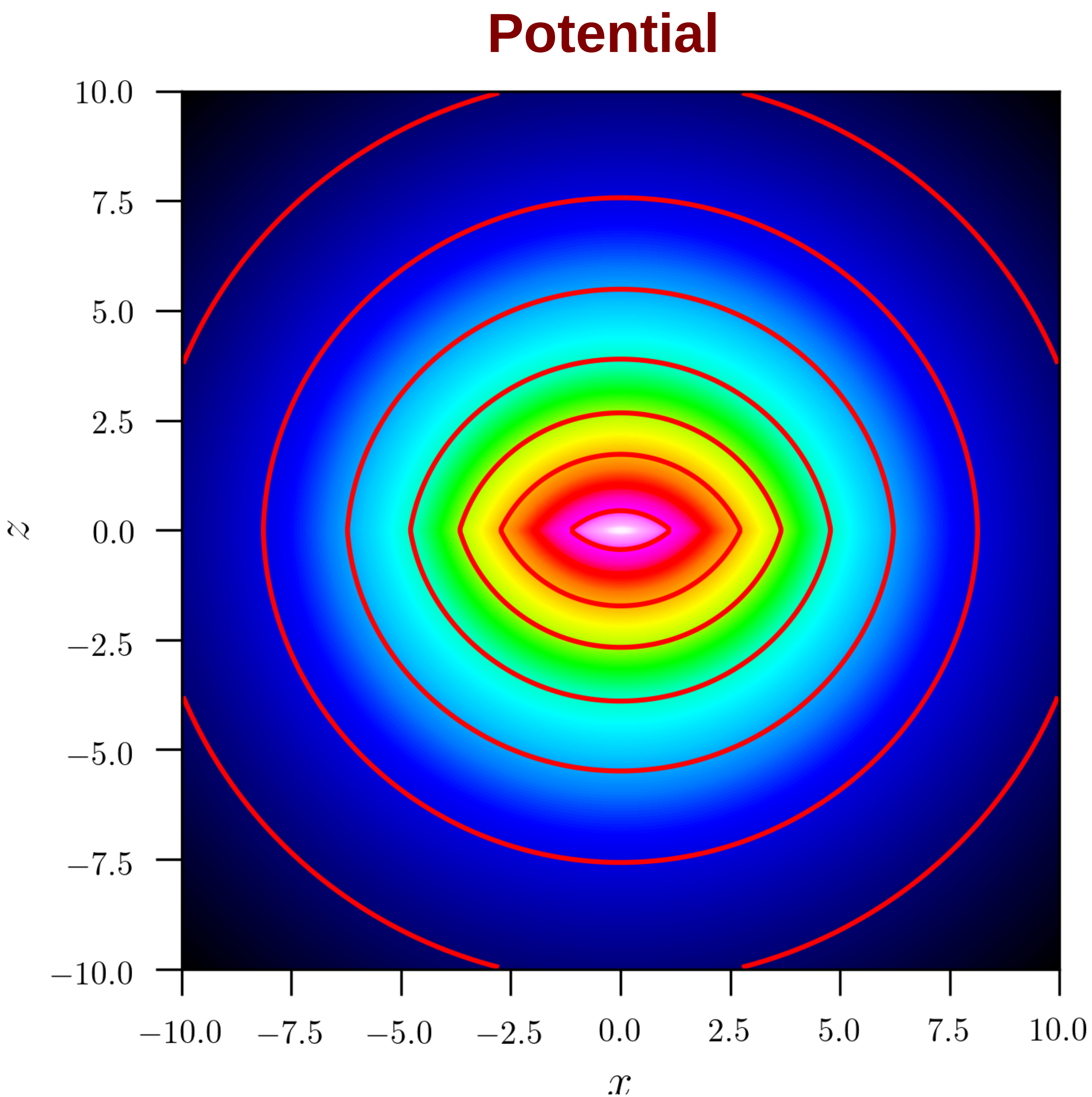
$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\nu+k)!} \left(\frac{1}{2}z\right)^{\nu+2k}$$

Exponential disk

Circular velocity rotation curve



Exponential disk



Mestel disk

$$\Sigma(R) = \begin{cases} \frac{v_0^2}{2\pi G R} & (R < R_{\max}) \\ 0 & (R \geq R_{\max}) \end{cases}$$

"2D" version of the Isothermal sphere

for $\mathrm{R_{max}} \to \infty$

$$v_c^2 = \frac{2v_0^2}{\pi} \int_0^R \frac{a}{\sqrt{R^2 - a^2}} = v_0^2 = \mathrm{cte}$$

EXERCICE

Computing the cumulative mass:

$$M(R) = 2\pi \int_0^R R' \, R' \, \Sigma(R') = \frac{v_0^2 R}{G}$$

we get:

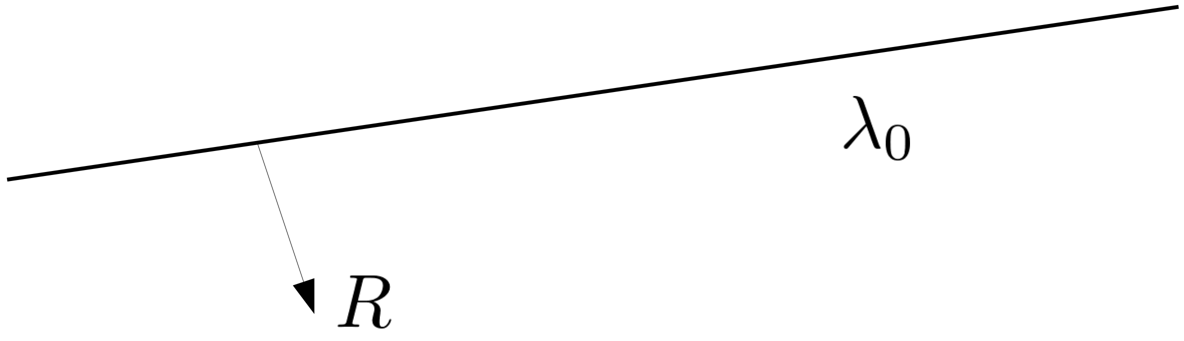
$$v_0^2 = v_c^2(R) = \frac{GM(R)}{R}$$

This is very specific to the Mestel disk…
In general the external mass matter.

Potential Theory

Ideal but useful models

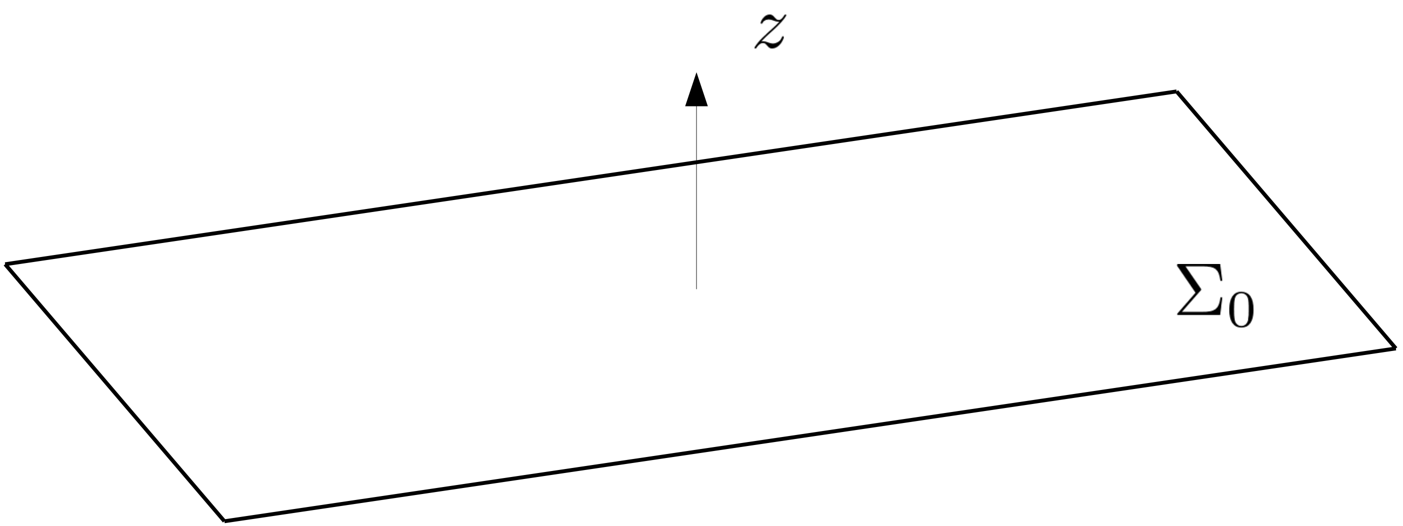
Potential of an infinite wire of constant linear density



$$\Phi(R) = 2\,G\,\lambda_0\,\ln(R) + C$$

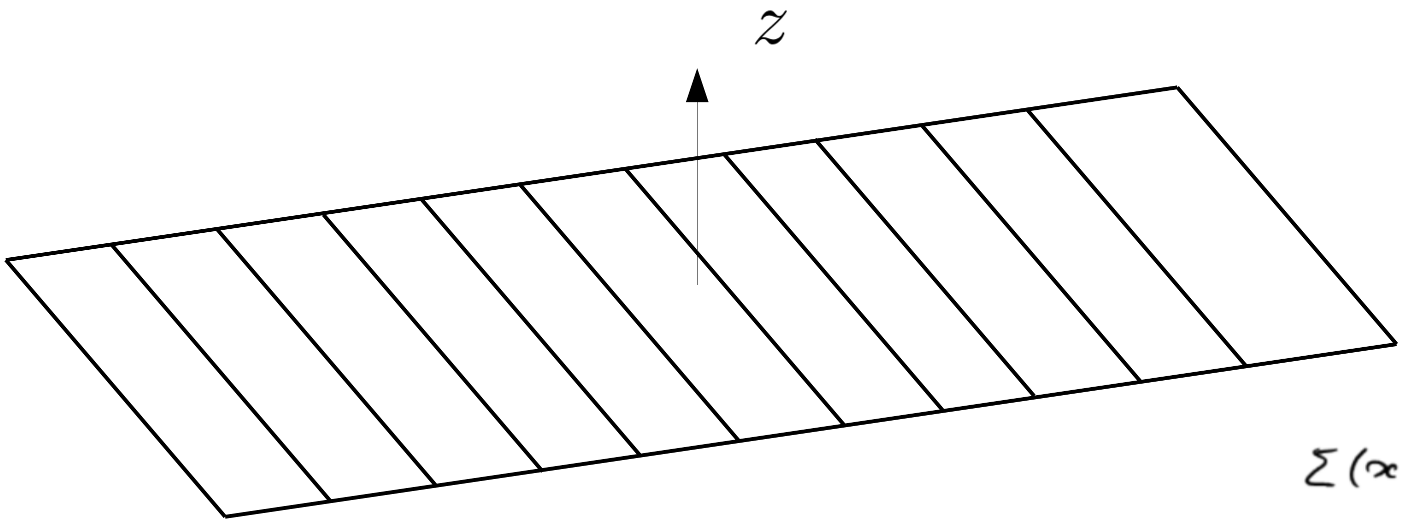
EXERCICE

Potential of an infinite slab of constant surface density



$$\Phi(z) = 2\pi G \, \Sigma_0 \, |z| + C$$

Potential of an infinite slab with an oscillatory surface density



$$k = |\vec{k}| = \frac{2\pi}{\lambda}$$

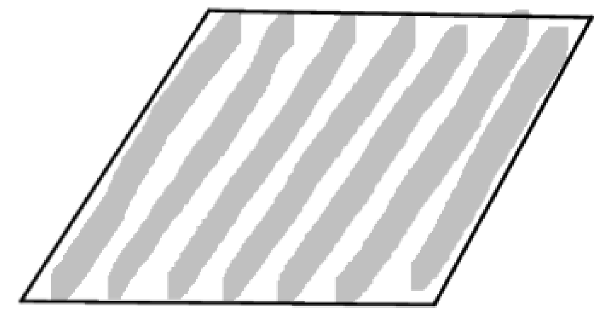
$$\Sigma(x, y) = \Sigma_1 \, \mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)} \right)$$

! will be negative !

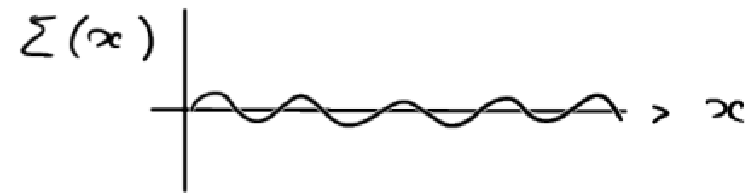
$$\Phi(x, y, z) = -\frac{2\pi G \, \Sigma_1}{|\vec{k}|} \mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)} \right) e^{-|\vec{k}| \, z}$$

Potential of an infinite slab with an oscillatory surface density

$$\Sigma(x,y) = \mathrm{Re}\left(\Sigma_0 \, e^{i(\vec{k}\vec{x})} \right)$$



Without loss of generality we can restrict to:



$$\Sigma(x) = \Sigma_0 \, e^{ikx}$$

Poisson equation $\qquad \vec{\nabla}^2 \phi(x,z) = 4\pi G \, \Sigma(x) \, \delta(z)$

Assume a corresponding potential of the type

$$\phi(x,z) = \phi_0 \, e^{ikx - |kz|}$$

Method : Integrate the Poisson equation over z

$$\vec{\nabla}^2 \phi = 4\pi G \rho$$

$$\int_{-\xi}^{\xi} dz \, \vec{\nabla}^2 \phi = \int_{-\xi}^{\xi} dz \, 4\pi G \rho$$

and take the limit $\xi \to 0$

$$\underbrace{\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, \vec{\nabla}^2 \phi}_{①} = \underbrace{\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, 4\pi G \rho}_{②}$$

COMPLEMENT

COMPLEMENT

(1) $$\nabla^2 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$\frac{\partial^2 \phi}{\partial x^2}(x, y, 0)$ is continuous across $z = 0$

$$\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, \frac{\partial^2 \phi}{\partial x^2}(x, y, z) = 0$$

$\lim_{\xi \to 0^+} = \lim_{\xi \to 0^-}$

$$\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, \frac{\partial^2 \phi}{\partial y^2}(x, y, z) = 0$$

$$\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \, \frac{\partial^2 \phi}{\partial z^2}(x, y, z) = \lim_{\xi \to 0} \left. \frac{\partial \phi}{\partial z} \right|_{-\xi}^{\xi}$$

$$= \lim_{\xi \to 0} \left. \phi_0 \, |k| \, \mathrm{sgn}(z) \, e^{ikx - |kz|} \right|_{-\xi}^{\xi} = -2|k| \, \phi_0 \, e^{ikx}$$

(2) $$\lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \; 4\pi G \rho = \lim_{\xi \to 0} \int_{-\xi}^{\xi} dz \; 4\pi G \Sigma_0 e^{ikx} \delta(z)$$

$$= 4\pi G \Sigma_0 e^{ikx}$$

Combining (1) and (2)

$$-2|k| \phi_0 e^{ikx} = 4\pi G \Sigma_0 e^{ikx}$$

$$\phi_0 = -\frac{2\pi G \Sigma_0}{|k|}$$

$$\boxed{\phi(x, y, z) = -\frac{2\pi G \Sigma_0}{|k|} e^{ikx - |kz|}}$$

COMPLEMENT

Thus for $\Sigma(x,y) = \Sigma_0 \, e^{i\vec{k}\vec{x}}$

$$\phi(x,y,z) = -\frac{2\pi G \Sigma_0}{|\vec{k}|} \, e^{i\vec{k}\vec{x} - |\vec{k}z|}$$

Note if the surface density evolves as a plane wave

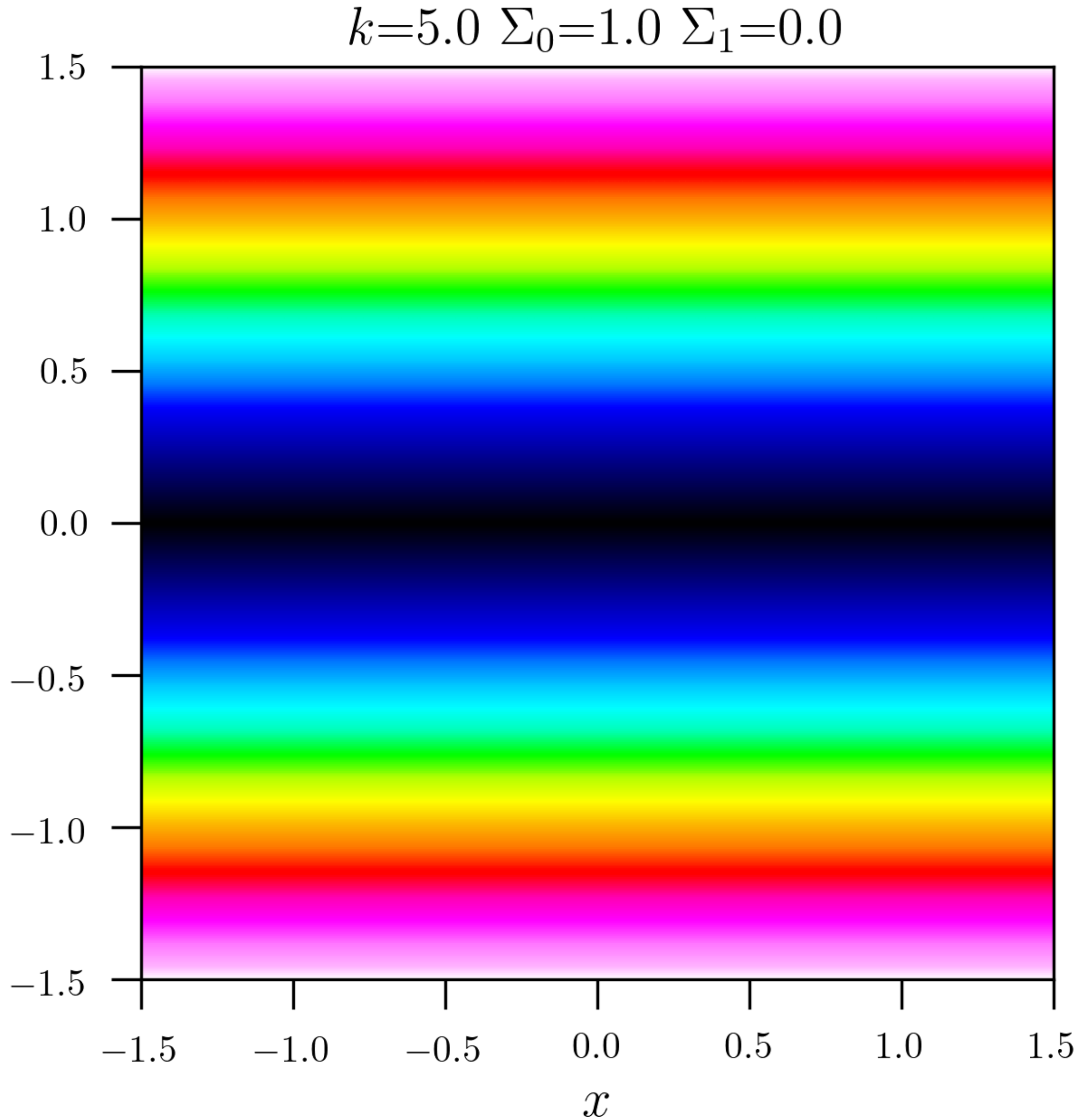
$$\Sigma(x,y,t) = \Sigma_0 \, e^{i(\vec{k}\vec{x} - \omega t)}$$

$$\phi(x,y,z,t) = -\frac{2\pi G \Sigma_0}{|\vec{k}|} \, e^{i(\vec{k}\vec{x} - \omega t) - |\vec{k}z|}$$

Potential of an Infinite slab

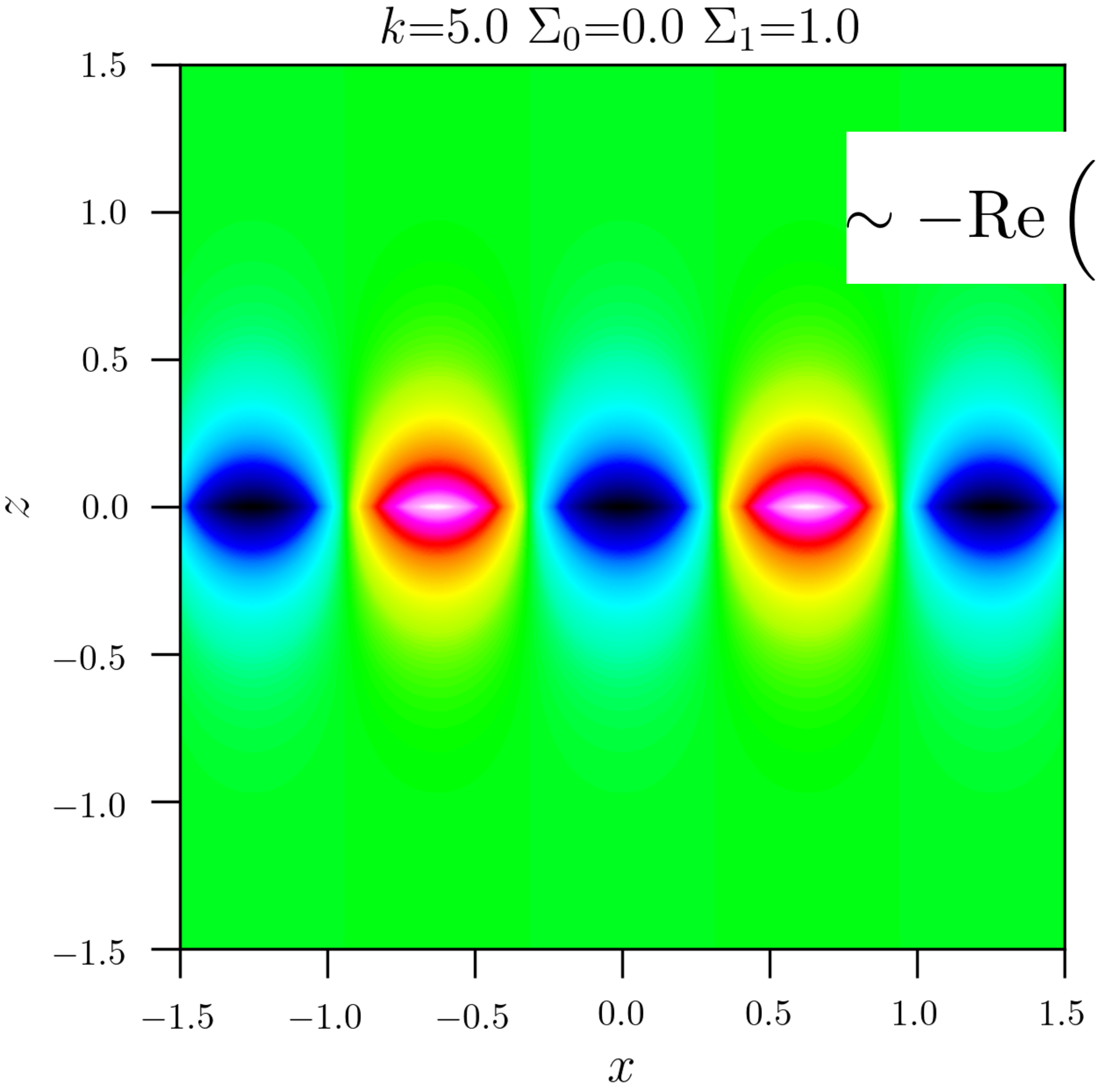
$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$

$\sim \Sigma_0 \, |z|$



Potential of an Infinite slab

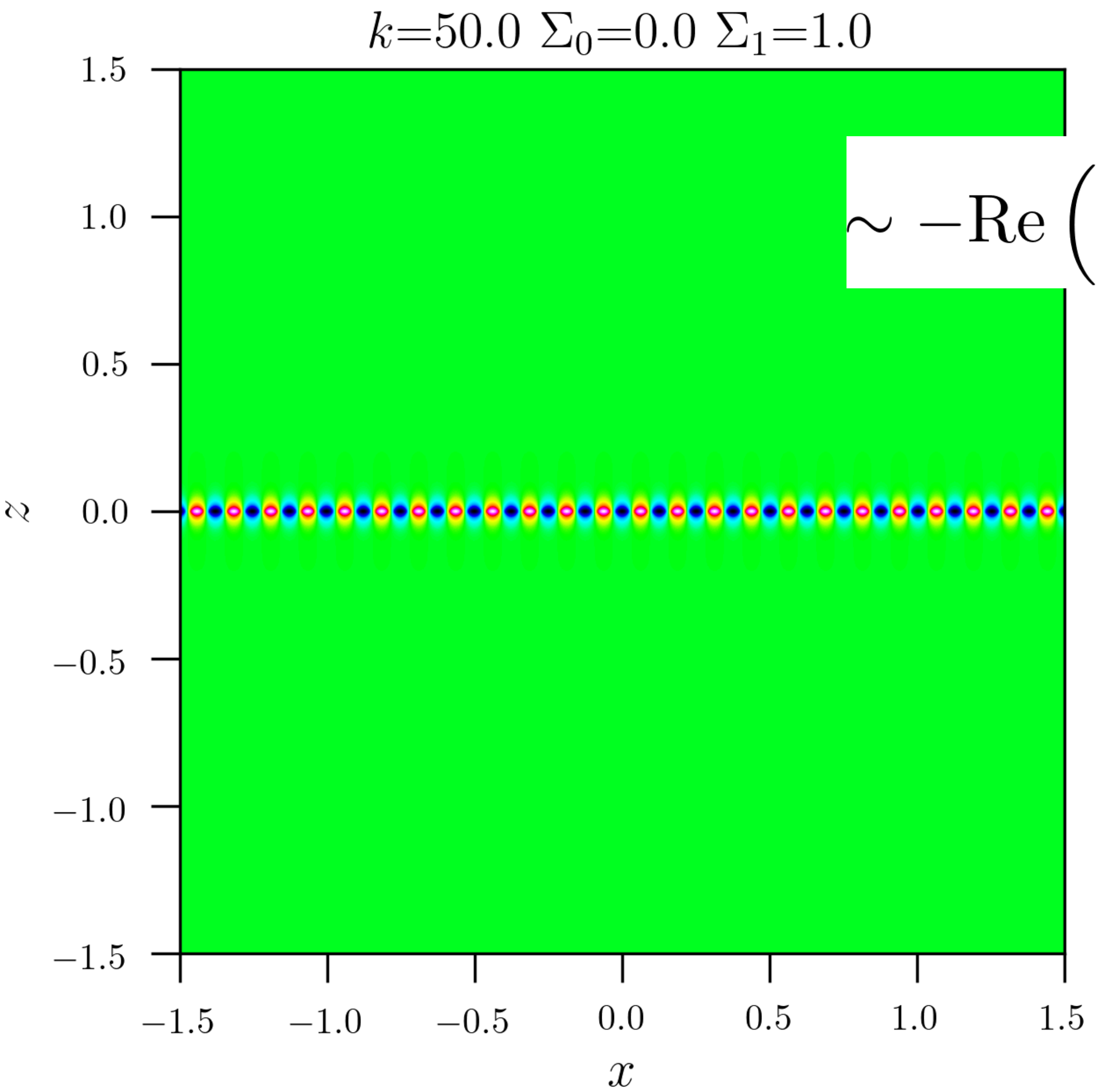
$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$



$$\sim -\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)\frac{e^{-k\,|z|}}{k}$$

Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$

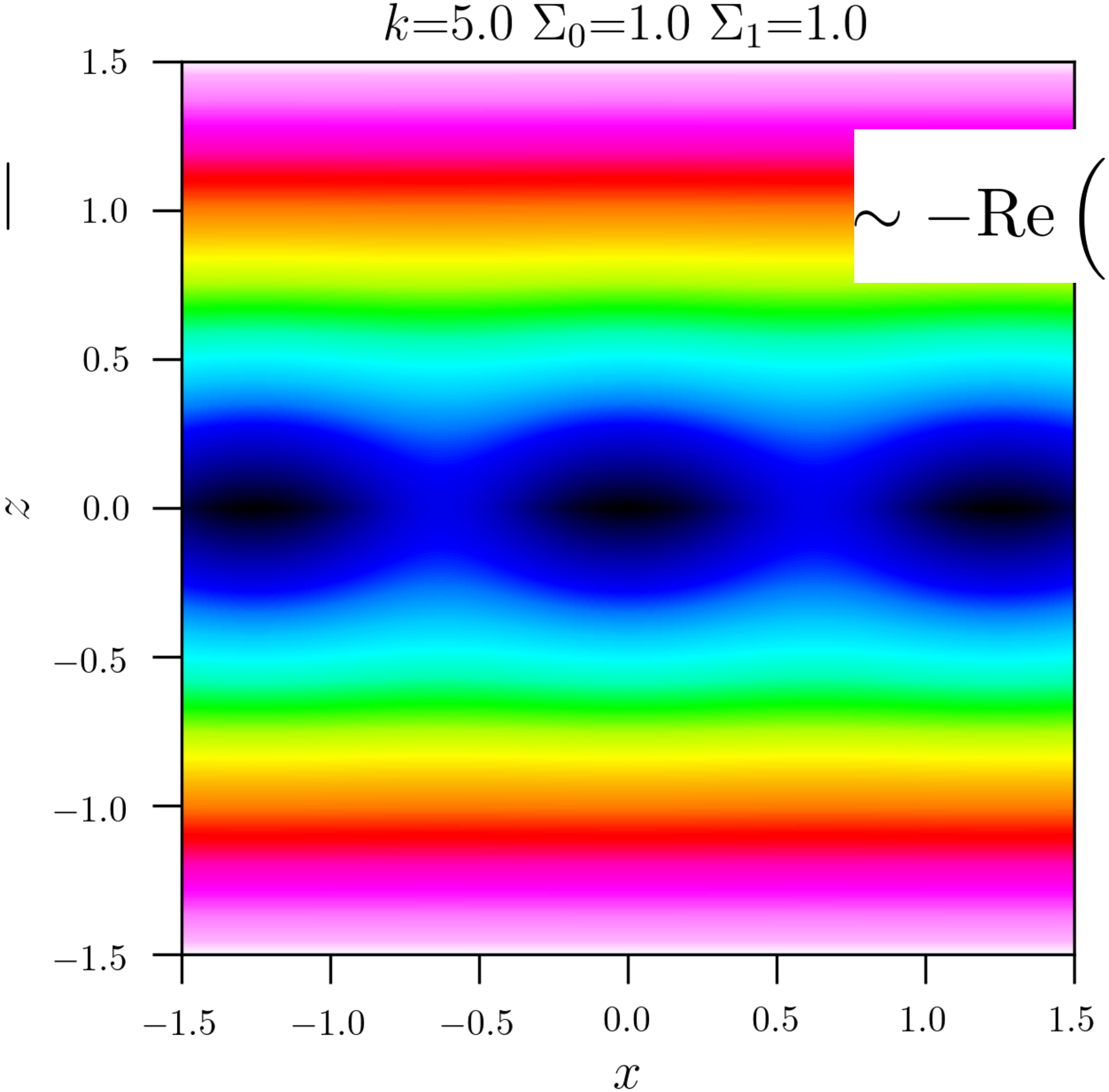


$$\sim -\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right)\frac{e^{-k\,|z|}}{k}$$

Potential of an Infinite slab

$$\Sigma(x) = \Sigma_0 + \Sigma_1 Re\left(e^{ikx}\right)$$

$\sim \Sigma_0 \, |z|$

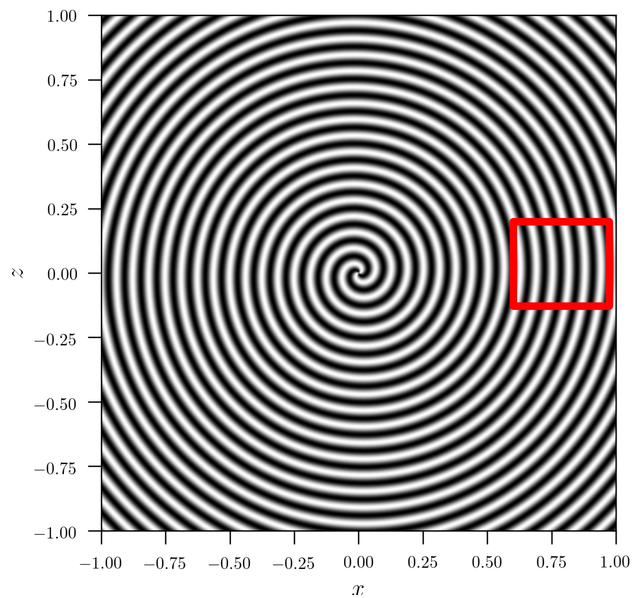


$$\sim -\mathrm{Re}\left(e^{i\left(\vec{k}\cdot\vec{x}\right)}\right) \frac{e^{-k\,|z|}}{k}$$

Potential of an infinite slab with a tightly wound spiral pattern

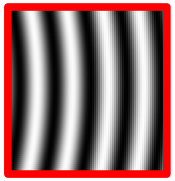
m=2

$$\Sigma(R,\phi) = H(R)\,\mathrm{Re}\left(e^{i[m\,\phi + f(R)]}\right)$$



if $$\left|\frac{\partial f}{\partial R}\cdot R\right| \ll 1$$

WKB approximation
(Wentzel,Kramers,Brillouin)

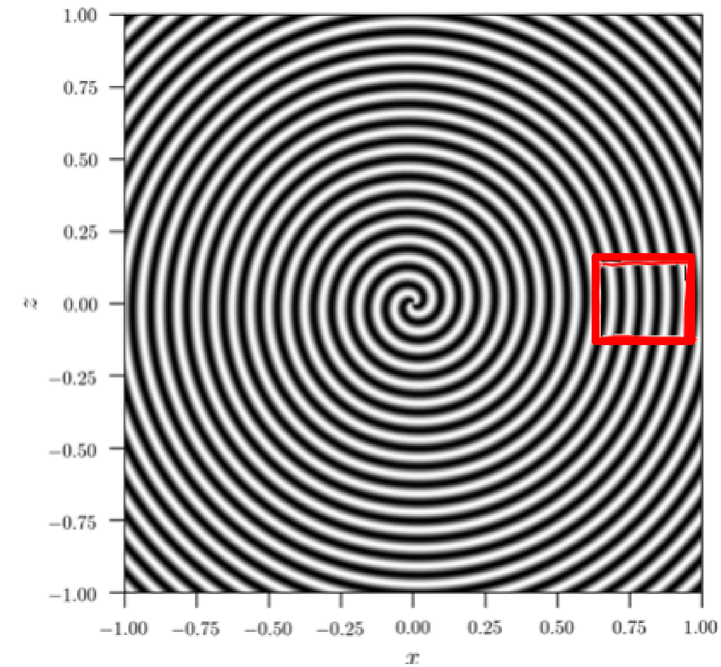


$$\Phi(R,\phi) = -\frac{2\pi G\,\Sigma_0}{\left|\frac{\partial f}{\partial R}\right|}\,H(R)\,\mathrm{Re}\left(e^{if(R)}\right)\,e^{-\left|\frac{\partial f}{\partial R}\cdot z\right|}$$

Potential of an infinite slab with a tightly wound spiral pattern

$m = 2$

$$\Sigma(R,\phi) = R_e\Big(\underbrace{H(R)}_{\text{slow variation}}\ \underbrace{e^{i(m\theta + g(R))}}_{\text{rapid variation}}\Big)$$

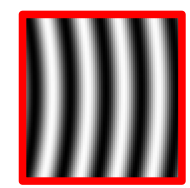


__Note__ $\boxed{m\theta + g(R) = \text{cte}}$

describe a spiral $\quad g(R)$ = shape function

__Idea__ : WKB approximation

far from the center, Σ is nearly $\sim e^{i(kx)}$



__Indeed__ Developping $g(R)$ around R_0 gives

$$g(R) \cong g(R_0) + \left.\frac{\partial g}{\partial R}\right|_{R_0} (R - R_0)$$

COMPLEMENT

For $\theta = 0$

$$\Sigma(R,\theta) = \underbrace{\mu(R_0)\, e^{i f(R_0)}}_{\text{no radial dependency}} \; \underbrace{e^{i \frac{\partial f}{\partial R}\big|_{R_0} (R-R_0)}}_{e^{ikx}}$$

$$e^{ikx} \quad \begin{cases} k = \frac{\partial f}{\partial R}\Big|_{R_0} \\ x = R - R_0 \end{cases}$$

We directely have the solution from the infinite slab

$$\phi(R,\theta) = -\frac{2\pi G}{\left|\frac{\partial f}{\partial R}\right|} \mu(R_0)\, e^{i f(R_0)}\, e^{i \frac{\partial f}{\partial R}\big|_{R_0}(R-R_0)}\, e^{-\left|\frac{\partial f}{\partial R} z\right|}$$

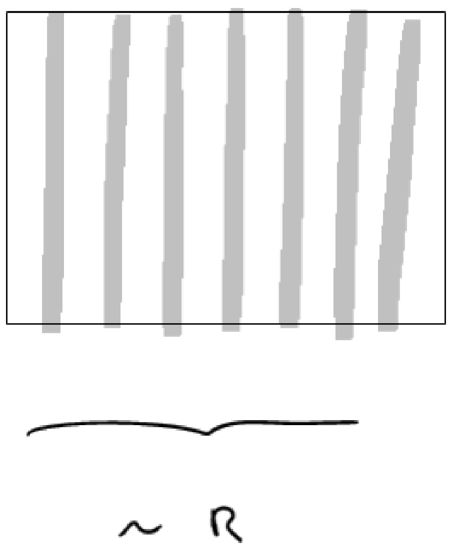
Choosing $R_0 = R$

$$\boxed{\phi(R,\theta) = -\frac{2\pi G}{\left|\frac{\partial f}{\partial R}\right|} \mu(R)\, e^{i f(R)}\, e^{-\left|\frac{\partial f}{\partial R} z\right|}}$$

COMPLEMENT

Validity of the approximation

- we want a large number of "oscillations" over a small radius compared to R



$$\left|\frac{\partial \delta}{\partial R}\right| \cdot R \gg 1$$

COMPLEMENT

Stellar orbits

1st part

Generalities

Stellar orbits

Why studying stellar orbits ?

- understand the motion of stars in stellar systems and galaxies

→ understand the observed kinematics
→ constraints the mass model
→ confirm the Newton law of gravity

We will assume :

- a smoothed gravitational field
- time independent potentials

Stellar orbits

Definitions

- trajectory — solution of the equation of motion

$$\ddot{\vec{x}} = -\vec{\nabla}\Phi(\vec{x})$$

defined on a finite interval:

$$\vec{x}(t), \vec{x}(t_0) = \vec{x_0}, t \in [t_0, t_1]$$

- orbit — a trajectory defined on an infinite time interval

$$\vec{x}(t), \vec{x}(t_0) = \vec{x_0}, t \in [-\infty, \infty[$$

- periodic orbit — a closed orbit

$$\forall\, t, \exists\, T, \vec{x}(t+T) = \vec{x}(t),\ \ \dot{\vec{x}}(t+T) = \dot{\vec{x}}(t)$$

- stationary point — a point such that:

$$\ddot{\vec{x}} = \dot{\vec{x}} = 0$$

The End