Solutions to Homework 4
CS-526 Learning Theory

Problem 1. Exercise 3 of Chapter 6
We prove that VC-dimension of $\mathcal{H}_{n-parity}$ is n. First, observe that $|\mathcal{H}_{n-parity}| = 2^n$ and it follows that VCdim$(H_{n-parity}) \leq \log(|H_{n-parity}|) = n$. Also VCdim$(H_{n-parity}) \geq n$ since it shatters the standard basis $\{e_i\}_{i=1}^n$, where e_i is a length-n vector that has 1 at position i and 0 everywhere else. To see that, observe that $h_J(e_j) = 1$ iff $i \in J$ and hence for any vector of labels $(y_1, \ldots, y_n)$ taking $J = \{i | y_i = 1\}$ will suffice.

Problem 2. VC dimension of circles
We will show that the VC dimension of $\mathcal{H}$ is $d = 3$.

1. Take three points in $\mathbb{R}^2$ located at the corners of an equilateral triangle. It is then clear that a circle can select any single one of these points, but also any pair of points and of course all three points together.

2. We show that $\mathcal{H}$ cannot shatter any set of 4 points. Consider 4 points A, B, C and D.

First, assume that one of the points is in the convex hull of the other 3 points. It is then impossible to label these 3 points with 1 while labeling the point in the convex hull with 0.

If no point is in the convex hull of the three other points then the 4 points form a convex quadrilateral $ABCD$. The line segment $[AC]$ is the first diagonal, and $[BD]$ is the second one. The two diagonals $[AC]$ and $[BD]$ must intersect each other.

We now claim that it is impossible to have circles such that the corresponding functions implement both $(0, 1, 0, 1)$ and $(1, 0, 1, 0)$. This is true since it is impossible to have two circles C_1 and C_2 such that

- C_1 contains only A and C, C_2 contains only B and D, and
- $[AC]$ cuts $[BD]$.

If such C_1 and C_2 existed, it would imply that $(C_1 \cup C_2) \setminus (C_1 \cap C_2)$ has 4 disjoint parts.

Below, we propose another way to show that the 4 points forming the convex quadrilateral $ABCD$ cannot be shattered by $\mathcal{H}$. This is a proof by contradiction. Suppose that $\mathcal{H}$ shatters the four points. The sum of the four interior angles is 360°. Without loss of generality, we have $\angle ABC + \angle CDA \geq 180°$. Because $\mathcal{H}$ shatters the four points, there is a circle $\mathcal{C}$ that contains A, C but not B, D. Let B', D' be the intersections of the line (BD) with $\mathcal{C}$, and let A', C' be the intersections of the line (AC) with $\mathcal{C}$. Clearly $[B'D'] \subset [BD]$ and $[AC] \subseteq [A'C']$. Note that

$$\angle A'B'C' + \angle C'D'A' > \angle ABC + \angle CDA \, .$$

Besides, the quadrilateral $A'B'C'D'$ is inscribed in the circle $\mathcal{C}$ so:

$$\angle A'B'C' + \angle C'D'A' = \angle B'C'D' + \angle D'A'B' = 180^\circ \ .$$

Hence the contradiction: $180^\circ = \angle A'B'C' + \angle C'D'A' > 180^\circ$.