Midterm exam

Please pay attention to the presentation of your answers! **(2 points)**

Exercise 1. Quiz. (15 points) Answer each yes/no question below (1 pt) and provide a short justification (proof or counter-example) for your answer (2 pts).

a) Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-field on $\mathbb{R}$. Recall that $\mathbb{Q}$ denotes the set of all rational numbers. Is $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$?

b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space. Let $X : \Omega \to \mathbb{R}$ be an $\mathcal{F}-$measurable random variable. Is $|X|$ also an $\mathcal{F}-$measurable random variable?

c) Is the converse of part b) true? That is, if $|X|$ is an $\mathcal{F}-$measurable random variable, then is X an $\mathcal{F}-$measurable random variable?

d) Let X be a Gaussian random vector which is known to have the covariance matrix

$$\mathrm{Cov}(X) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Is X a continuous random vector?

e) Let $U \sim \text{Uniform}[0,1]$ and define

$$X_n = n 1_{[0, \frac{1}{\sqrt{n}}]}(U), \quad n = 1, 2, \ldots$$

Does X_n converge in probability to zero?

Exercise 2. (15 points)

Let Ω be an arbitrary set and $\mathcal{F}$ be a $\sigma-$field on Ω. In this problem we will show that if $\mathcal{F}$ is infinite, it must be uncountable. We will proceed with proof by contradiction and assume that $\mathcal{F}$ is countable.

a) For every $\omega \in \Omega$, define $B_\omega = \bigcap_{A \in \mathcal{F}:\, \omega \in A} A$. Is $B_\omega \in \mathcal{F}$? Why or why not?

b) Let $\mathcal{C} = \{B_\omega\}_{\omega \in \Omega}$ be a collection of all such unique B_ω. Argue that $\mathcal{C}$ partitions Ω and that it is at most finite, or countable.

c) Argue that $\sigma(\mathcal{C}) = \mathcal{F}$. That is, the σ-field generated by $\mathcal{C}$ is exactly $\mathcal{F}$.

d) Conclude from parts (a) - (c) that there is a contradiction and it is not possible for $\mathcal{F}$ to be countable.

Exercise 3. (14 points)

The moment-generating function of a random variable X is defined for any $t \in \mathbb{R}$ as

$$M_X(t) = \mathbb{E}\left(e^{tX}\right).$$

(Notice that it is similar but not equal to the characteristic function of X!) Let $X \sim \mathrm{Bi(n, p)}$ where, recall that, the Binomial distribution with parameters (n, p) measures the probability of k successes in n independent Bernoulli trials each with parameter p.

a) Prove that for every $a \in \mathbb{R}$ and $t > 0$,

$$\mathbb{P}(X \geq a) \leq e^{-ta} M_X(t).$$

b) Show that

$$M_X(t) = (pe^t + (1-p))^n.$$

c) Using the inequality in part a) and optimizing over all $t > 0$, show that for any fixed q such that $p < q < 1$,

$$\mathbb{P}(X \geq qn) \leq \left(\frac{p}{q}\right)^{qn} \left(\frac{1-p}{1-q}\right)^{(1-q)n}.$$

d) Using Markov inequality, show that

$$\mathbb{P}(X \geq qn) \leq \frac{p}{q}$$

and compare this inequality with the one in part c).

Exercise 4. (14 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\Omega = \{(\omega_1, \omega_2) \colon \omega_1, \omega_2 \in \{1, 2, \ldots, n\}\}$ for some $n \geq 1$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\omega_1, \omega_2) = \frac{1}{n^2}$ for all $(\omega_1, \omega_2) \in \Omega$.

a) Let $X_1 = \omega_1 + \omega_2$. Describe $\sigma(\{X_1\})$, the σ-field generated by X_1. How many atoms does it have? What are they?

b) Let $X_2 = \omega_1 - \omega_2$. Are X_1 and X_2 independent? Why or why not?

c) Let $X = \omega_1$, $Z = 1_{\{\omega_1 = \omega_2\}}$, and $Y = 1_{\{\omega_1 + \omega_2 = n+1\}}$. Are X, Y, Z pairwise independent? Why or why not?