Midterm exam: solutions

Please pay attention to the presentation of your answers! **(2 points)**

Advanced Probability and Applications EPFL - Spring Semester 2023-2024

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Exercise 1. Quiz. (15 points) Answer each yes/no question below (1 pt) and provide a short justification (proof or counter-example) for your answer (2 pts).

a) Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-field on $\mathbb{R}$. Recall that $\mathbb{Q}$ denotes the set of all rational numbers. Is $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$?

Answer: Yes. Every singleton $\{x\}, x \in \mathbb{R}$ belongs to $\mathcal{B}(\mathbb{R})$. Also, since $\mathcal{B}(\mathbb{R})$ is a σ-field, every countable union of sets in $\mathcal{B}(\mathbb{R})$ also belongs to $\mathcal{B}(\mathbb{R})$. Since $\mathbb{Q}$ is a countable union of real numbers, $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$.

b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space. Let $X : \Omega \to \mathbb{R}$ be an $\mathcal{F}-$measurable random variable. Is $|X|$ also an $\mathcal{F}-$measurable random variable?

Answer: Yes. The function $g(x) = |x|$ is continuous and therefore it is Borel-measurable. Since X is $\mathcal{F}$-measurable and g is Borel-measurable, then $g(X) = |X|$ is also $\mathcal{F}$-measurable.

c) Is the converse of part b) true? That is, if $|X|$ is an $\mathcal{F}-$measurable random variable, then is X an $\mathcal{F}-$measurable random variable?

Answer: No. For example, let $\Omega = \{-2, -1, 1, 2\}$, $\mathcal{F} = \sigma(\{\{-2, -1\}, \{1\}, \{2\}\})$, and $X(\omega) = \omega$. Then, $|X|$ is $\mathcal{F}$-measurable, but X is not, since the set $\{X = -1\} = \{-1\}$ does not belong to $\mathcal{F}$.

d) Let X be a Gaussian random vector which is known to have the covariance matrix

$$\mathrm{Cov}(X) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Is X a continuous random vector?

Answer: No. The covariance matrix $\mathrm{Cov}(X)$ is not invertible, and so X is not a continuous vector. For example $X = (X_1, X_2, X_1 + X_2)$ where $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$ could be such a vector. In particular, it will be supported on a hyperplane in 3D space which has Lebesgue measure zero.

e) Let $U \sim \text{Uniform}[0, 1]$ and define

$$X_n = n 1_{[0, \frac{1}{\sqrt{n}}]}(U), \quad n = 1, 2, \ldots$$

Does X_n converge in probability to zero?

Answer: Yes. Observe that for any $\epsilon > 0$

$$\mathbb{P}\left(|X_n| \geq \epsilon\right) \leq \mathbb{P}\left(U \leq \frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}} \to 0.$$

Exercise 2. (15 points)

Let Ω be an arbitrary set and $\mathcal{F}$ be a $\sigma-$field on Ω. In this problem we will show that if $\mathcal{F}$ is infinite, it must be uncountable. We will proceed with proof by contradiction and assume that $\mathcal{F}$ is countable.

a) For every $\omega \in \Omega$, define $B_\omega = \bigcap_{A \in \mathcal{F}:\, \omega \in A} A$. Is $B_\omega \in \mathcal{F}$? Why or why not?

Answer: We have assumed that $\mathcal{F}$ is countable. Thus, the collection of all the sets containing ω i.e., $S_\omega = \{A : \omega \in A\}$ can be at most countable, as $S_\omega \subset \mathcal{F}$. Further, note that the countable intersection of sets in $\mathcal{F}$ is also an element of $\mathcal{F}$. Thus, $B_\omega := \cap S_\omega$ is an element of $\mathcal{F}$.

b) Let $\mathcal{C} = \{B_\omega\}_{\omega \in \Omega}$ be a collection of all such unique B_ω. Argue that $\mathcal{C}$ partitions Ω and that it is at most finite, or countable.

Answer: To show that B_ω partitions $\mathcal{F}$ we need to show that: 1)$\forall \omega_1, \omega_2 \in \Omega$, we have $B_{\omega_1} \cap B_{\omega_2} = \emptyset$ or $B_{\omega_1} = B_{\omega_2}$, 2) that $\cup_{\omega \in \Omega} B_\omega = \Omega$.

1) Suppose there exists $\omega_2 \in B_{\omega_1}$ such that $B_{\omega_1} \neq B_{\omega_2}$. Then, $B_{\omega_1} \cap B_{\omega_2}$ is a strict subset of B_{ω_2} or it is exactly B_{ω_2}. In the first case, it contradicts the fact that B_{ω_2} is the smallest set in $\mathcal{F}$ containing ω_2. In the second case, it means that B_{ω_2} is a proper subset of B_{ω_1} which again contradicts the fact that B_{ω_1} is the smallest set in $\mathcal{F}$ containing ω_1. Indeed, either $\omega_1 \in B_{\omega_2}$ or $\omega_1 \in B_{\omega_1} \cap B^c_{\omega_2}$.

2) Since every $\omega \in \Omega$ is in some B_ω, $\cup_{\omega \in \Omega} B_\omega = \Omega$.

Since $\mathcal{F}$ is countable, and $\mathcal{C}$ is a subset of $\mathcal{F}$ it is either countable or finite.

c) Argue that $\sigma(\mathcal{C}) = \mathcal{F}$. That is, the σ-field generated by $\mathcal{C}$ is exactly $\mathcal{F}$.

Answer:

For any $A \in \mathcal{F}$ we can show that $A = \cup_{\omega \in A} B_\omega$. Indeed, $A \subset \cup_{\omega \in A} B_\omega$ is trivial. We can show that $\cup_{\omega \in A} B_\omega \subset A$ by a similar argument as in part b). Assume that there exists $\omega_1 \in \cup_{\omega \in A} B_\omega$ such that $\omega_1 \notin A$. But then, either $B_{\omega_1} \cap A = \emptyset$ or $B_{\omega_1} \cap A$ is a proper subset of B_{ω_1} which again contradicts the minimality of B_{ω_1} for some $\omega_2 \in B_{\omega_1} \cap A$.

d) Conclude from parts (a) - (c) that there is a contradiction and it is not possible for $\mathcal{F}$ to be countable.

Answer: Observe that we have shown that $\mathcal{C}$ is exactly the set of atoms that generates $\mathcal{F}$ and that it is either finite or countable. By part b), a union of any subcollection of $\mathcal{C}$ produces a distinct subset of $\mathcal{F}$. Thus, if $\mathcal{C}$ is finite, it's power set is also finite. If $\mathcal{C}$ is countable, its power set is uncountable (See PSET 1, exercise 1). Either way, this contradicts the original assumption.

Exercise 3. (14 points)

The moment-generating function of a random variable X is defined for any $t \in \mathbb{R}$ as

$$M_X(t) = \mathbb{E}\left(e^{tX}\right).$$

(Notice that it is similar but not equal to the characteristic function of X!) Let $X \sim \mathrm{Bi(n,p)}$ where, recall that, the Binomial distribution with parameters (n, p) measures the probability of k successes in n independent Bernoulli trials each with parameter p.

a) Prove that for every $a \in \mathbb{R}$ and $t > 0$,

$$\mathbb{P}(X \geq a) \leq e^{-ta} M_X(t).$$

Answer: The result follows directly from the Chebyshev-Markov inequality with $\psi(x) = e^{tx}$.

b) Show that

$$M_X(t) = (pe^t + (1-p))^n.$$

Answer: We can write $X = \sum_{i=1}^{n} B_i$, where the B_i's are n iid Bernoulli(p) random variables. Then, for each B_i we have

$$\mathbb{E}(e^{tB_i}) = pe^t + 1 - p$$

so that we have

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) \\ &= \mathbb{E}(e^{t\sum B_i}) \\ &= \mathbb{E}\left(\prod_i e^{tB_i}\right) \\ &= \prod_i \mathbb{E}(e^{tB_i}) \\ &= (pe^t + 1 - p)^n. \end{aligned}$$

c) Using the inequality in part a) and optimizing over all $t > 0$, show that for any fixed q such that $p < q < 1$,

$$\mathbb{P}(X \geq qn) \leq \left(\frac{p}{q}\right)^{qn} \left(\frac{1-p}{1-q}\right)^{(1-q)n}.$$

Answer: By applying the inequality in part 1 to X with $a = qn$, we get

$$\mathbb{P}(X \geq gn) \leq \left(\frac{pe^t + 1 - p}{e^{tq}}\right)^n$$

Since y^n is an increasing function for $y > 0$, in order to optimize the right-hand side over t, we can substitute $z = e^t$ and optimize the function

$$\frac{pz + 1 - p}{z^q}$$

over $z > 0$. By taking the derivative and putting it equal to 0, we get

$$\frac{pz^q - qz^{q-1}(pz + 1 - p)}{z^{2q}} = 0 \iff pz - pqz - q(1-p) = 0 \iff z = \frac{q}{p} \cdot \frac{1-p}{1-q}.$$

Substituting $z = e^t$ in the right-hand side of the inequality leads to the result.

d) Using Markov inequality, show that

$$\mathbb{P}(X \geq qn) \leq \frac{p}{q}$$

and compare this inequality with the one in part c).

Answer: We have that

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_i B_i\right) = \sum_i \mathbb{E}(B_i) = np$$

so that Markov inequality for $a = qn$ becomes

$$\mathbb{P}(X \geq qn) \leq \frac{\mathbb{E}(X)}{nq} = \frac{np}{nq} = \frac{p}{q}.$$

Note that the second inequality does not depend on n. This is in general bad. In fact, when n is large we expect X to concentrate around np (its expectation). Since $q > p$, we therefore expect that $\mathbb{P}(X \geq qn) \to 0$ when $n \to \infty$. This is indeed what we get from the first inequality: the right-hand side goes to 0 when $n \to \infty$. However, the second inequality is just a constant for every n, and therefore it is very loose when n is large.

Exercise 4. (14 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\Omega = \{(\omega_1, \omega_2) \colon \omega_1, \omega_2 \in \{1, 2, \ldots, n\}\}$ for some $n \geq 1$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\omega_1, \omega_2) = \frac{1}{n^2}$ for all $(\omega_1, \omega_2) \in \Omega$.

a) Let $X_1 = \omega_1 + \omega_2$. Describe $\sigma(\{X_1\})$, the σ-field generated by X_1. How many atoms does it have? What are they?

Answer: The atoms of $\sigma(\{X_1\})$ have the form $S_j = \{w_1, w_2 \colon w_1 + w_2 = j\}$ for $j = 2, \ldots, 2n$. Thus, it has $2n - 1$ atoms, and consists of 2^{2n-1} subsets generated by every possible union of these atoms.

b) Let $X_2 = \omega_1 - \omega_2$. Are X_1 and X_2 independent? Why or why not?

Answer: No, X_1 and X_2 are not independent. For example,

$$\mathbb{P}\left(X_1 = 2, X_2 = 0\right) = \mathbb{P}\left(\{(\omega_1, \omega_2) = (1, 1)\}\right) = \frac{1}{n^2}.$$

On the other hand

$$\mathbb{P}\left(X_1 = 2\right) \mathbb{P}\left(X_2 = 0\right) = \frac{1}{n^2} \cdot \frac{1}{n}.$$

c) Let $X = \omega_1$, $Z = 1_{\{\omega_1 = \omega_2\}}$, and $Y = 1_{\{\omega_1 + \omega_2 = n+1\}}$. Are X, Y, Z pairwise independent? Why or why not?

Answer: It is always true that 1) $X \perp\!\!\!\perp Z$ and $X \perp\!\!\!\perp Y$. 2) For n even Z and Y are not independent. 3) For n odd, we also have that $Z \perp\!\!\!\perp Y$.

1) $X \perp\!\!\!\perp Z$:

$$\mathbb{P}(X = j, Z = 1) = \mathbb{P}(\{(\omega_1, \omega_2) = (j, j)\}) = \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} = \mathbb{P}(X = j)\,\mathbb{P}(Z = 1)$$

and

$$\mathbb{P}(X = j, Z = 0) = \mathbb{P}(\{(\omega_1, \omega_2) = (j, k) \colon k \neq j\}) = \frac{n-1}{n^2} = \frac{1}{n} \cdot \frac{n-1}{n} = \mathbb{P}(X = j)\,\mathbb{P}(Z = 0)$$

Note that $X \perp\!\!\!\perp Y$ follows by a completely symmetric argument.

2) For n odd Z and Y are not independent. We have

$$\mathbb{P}(Z = 1, Y = 1) = 0 \neq \frac{1}{n} \cdot \frac{1}{n} = \mathbb{P}(Z = 1)\,\mathbb{P}(Y = 1)$$

3) For n odd, we also have that $Z \perp\!\!\!\perp Y$:

$$\mathbb{P}(Z = 1, Y = 1) = \mathbb{P}\left(\left\{(\omega_1, \omega_2) = \left(\frac{n+1}{2}, \frac{n+1}{2}\right)\right\}\right) = \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} = \mathbb{P}(Z = 1)\,\mathbb{P}(Y = 1)$$

also

$$\mathbb{P}(Z = 0, Y = 0) = \frac{n^2 - 2n + 1}{n^2} = \frac{n-1}{n} \cdot \frac{n-1}{n} = \mathbb{P}(Z = 0)\,\mathbb{P}(Y = 0)$$

and

$$\mathbb{P}(Z = 1, Y = 0) = \mathbb{P}\left(\left\{(\omega_1, \omega_2) = (j, j), j \neq \frac{n+1}{2}\right\}\right) = \frac{n-1}{n^2} = \frac{1}{n} \cdot \frac{n-1}{n} = \mathbb{P}(Z = 1)\,\mathbb{P}(Y = 0).$$

Finally, the case with $\mathbb{P}(Z = 0, Y = 1)$ follows by symmetry.