Stellar orbits

3rd part

Outlines

Motions of stars in the Sun neighbourhood

- The Oort constants
- Epicycle frequences

Surfaces of section

- Integral of motions
- Poincaré maps

Motions of stars in the Sun neighbourhood

The Oort constants

Motions of stars in the neighbourhood of the Sun

- How can we learn about the global motions of stars in the Milky Way?

extent of inner halo
75 kly
M75
zone of avoidance
Pal 5
NGC 5634
co-moving at 630 km/s in the direction of the Great Attractor
60 kly
NGC 6356
0° galactic longitude
NGC 6284
Djor 1
NGC 6441
NGC 6316
M72
Pal 11
Pal 8
NGC 6453
NGC 6496
NGC 5897
45 kly
Centaurus Arm
NGC 6569
NGC 6380
NGC 6388
NGC 6584
Pal 12
NGC 6638
NGC 6652
45°
NGC 6517
NGC 6287
NGC 5986
315°
NGC 6101
NGC 6934
Norma/Outer Arm
Perseus Arm
Sagittarius Arm
Far 3kpc Arm
M70
molecular hydrogen ring
NGC 6139
NGC 1735
30 kly
NGC 5946
M14
mini spiral arm
Sgr A
NGC 6541
Near 3kpc Arm
Norma Arm
Lynga 7
NGC 5927
NGC 5286
NGC 7492
NGC 6539
nucleus
Ter 2
NGC 6558
SMC direction
bar
bulge
Ter9
AL3
Ter6
Ton2
NGC 6366
FSR 1716
Centaurus Arm
Sagittarius Arm
M2
NGC 6712
Djor2
M62
NGC 6352
M56
M15
NGC 6760
NGC 6749
NGC 6535
Sagitta
Vulpecula
Aquila
Scutum
Sagittarius
Scorpius
15 kly
Norma
Circinus
Centaurus
Quadrant IV
Quadrant I
W51
naked eye's limit
M68
Carina
Omega Centauri
LMC direction
NGC 362
M92
M13
M71
Cygnus X
NGC 6397
Cygnus Spur
Cygnus
Gould Belt
Sun
NGC 6752
47 Tuc
Vela
E3
NGC 2808
Andromeda Galaxy direction
90°
270°
Pleiades
M3
Orion Nebula
Vela molecular ridge
moving at 552 km/s with respect to the CMB
Quadrant II
Cepheus
NGC 288
Orion Spur
Perseus transit
Puppis
Perseus Arm
Rosette Nebula
NGC 4147
Canis Major
Quadrant III
NGC 1261
Norma/Outer Arm
Cassiopeia
Camelopardalis
Perseus
Auriga
Taurus
Orion
Monocerus
135°
225°
NGC 1851
NGC 2298
180°
Sagittarius Stream
globular clusters
gamma ray bubble
Helmi stream
open clusters
warped disk
Sun
emission nebula
inner halo
bulge
LMC
SMC
outer halo
Magellanic stream
0 1 2 3
kiloparsecs (kpc)
0 5 10
kilolight-year (kly)

Motions of stars in the neighbourhood of the Sun

- How can we learn about the global motions of stars in the Milky Way?

- Problem : we are living around the Sun, which is also moving around the Milky Way center ...

- Solution :
  - describe in a general framework the motions of nearby stars
  - deduce from observations of nearby stars global motions of the Milky Way

Taylor expansion of the velocity field a round $\vec{x}_0$

$$\vec{V}(\vec{x}) = \vec{V}_0 + \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} \end{pmatrix}_{\vec{x}_0} (\vec{x} - \vec{x}_0)$$

Jacobian matrix

$\vec{V}_0 = \vec{V}(x_0)$

Relative velocity field $\quad \delta\vec{V}(\vec{x}) = \vec{V}(\vec{x}) - \vec{V}_0$

$$\begin{cases} \delta V_x = \frac{\partial V_x}{\partial x}(x - x_0) + \frac{\partial V_x}{\partial y}(y - y_0) \\ \delta V_y = \frac{\partial V_y}{\partial x}(x - x_0) + \frac{\partial V_y}{\partial y}(y - y_0) \end{cases}$$

$$\boxed{\delta\vec{V} = J \cdot \delta\vec{x}}$$

$$J = \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} \end{pmatrix}$$

$$\delta\vec{x} = \vec{x} - \vec{x}_0$$

The matrix J can be decomposed on a basis

$$J = A \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + C \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + K \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

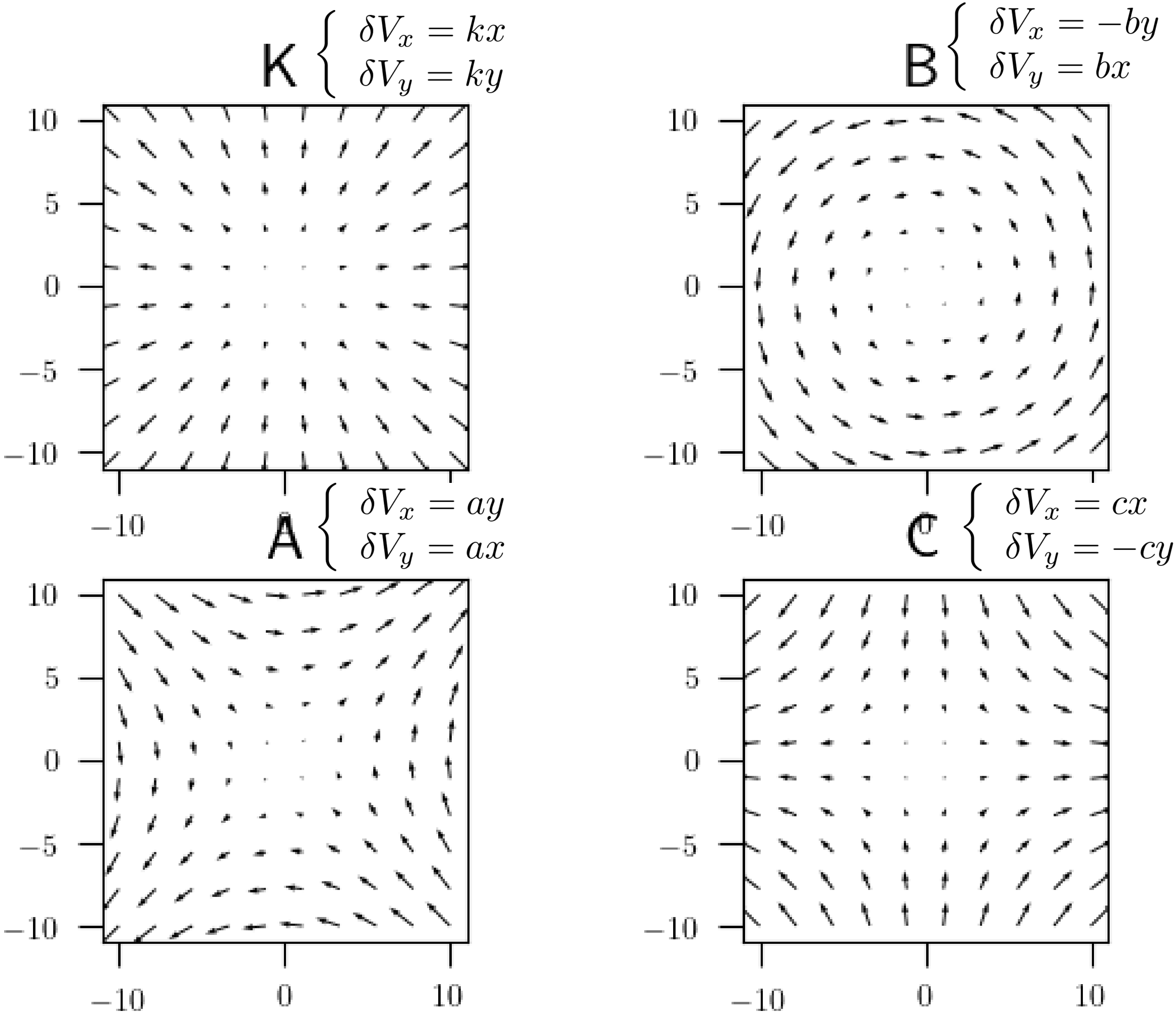
Interpretation

- "A" : shear $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- "B" : vorticity $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

- "C" : shear $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- "K" : divergeance $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



K $\begin{cases} \delta V_x = kx \\ \delta V_y = ky \end{cases}$
B $\begin{cases} \delta V_x = -by \\ \delta V_y = bx \end{cases}$
A $\begin{cases} \delta V_x = ay \\ \delta V_y = ax \end{cases}$
C $\begin{cases} \delta V_x = cx \\ \delta V_y = -cy \end{cases}$
10
5
0
−5
−10
−10
0
10

So, we have

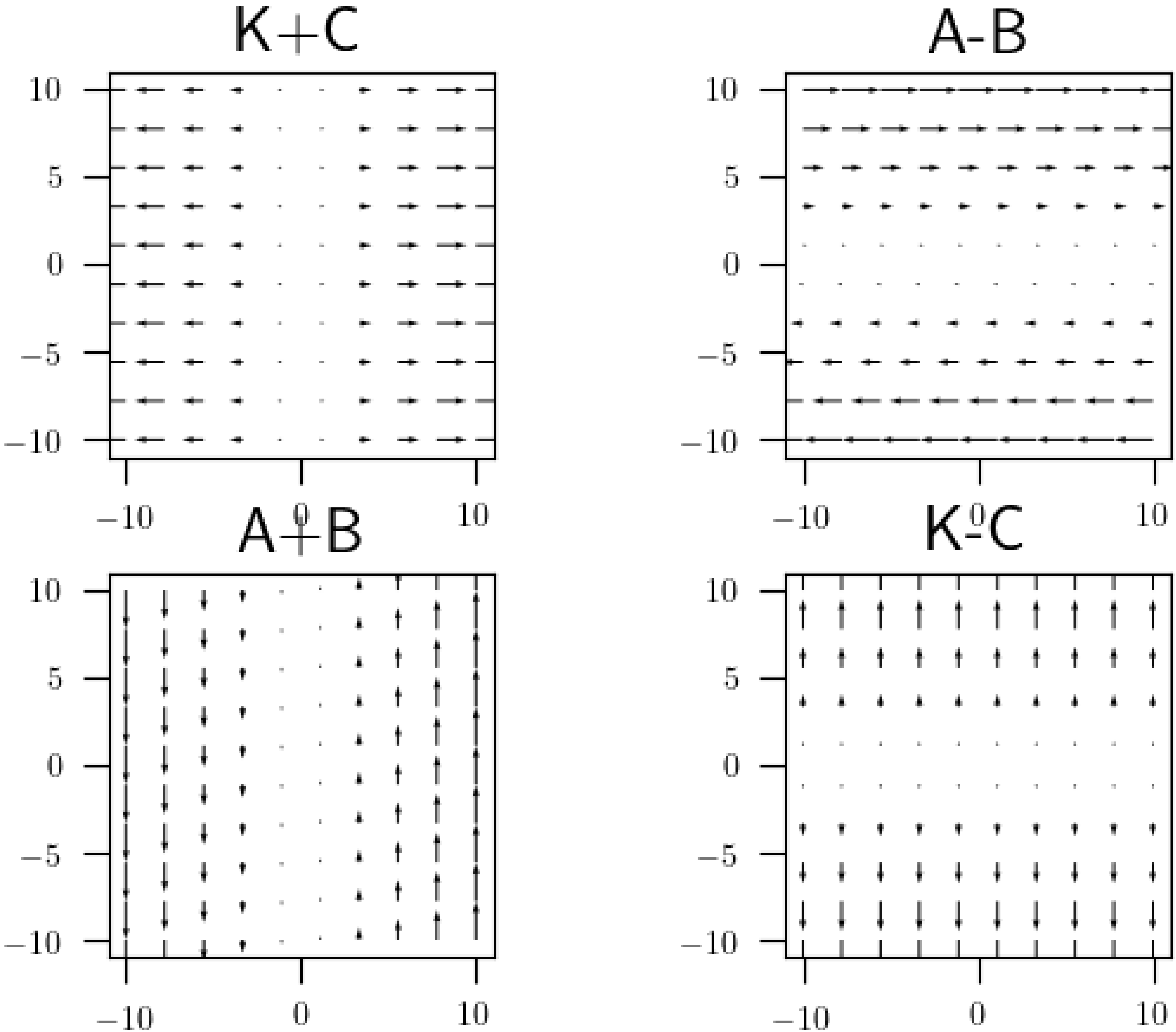
$$J = \begin{pmatrix} K + C & A - B \\ A + B & K - C \end{pmatrix} := \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} \end{pmatrix}$$

<u>A, B, C, K</u>

The Oort constants describe the local velocity field

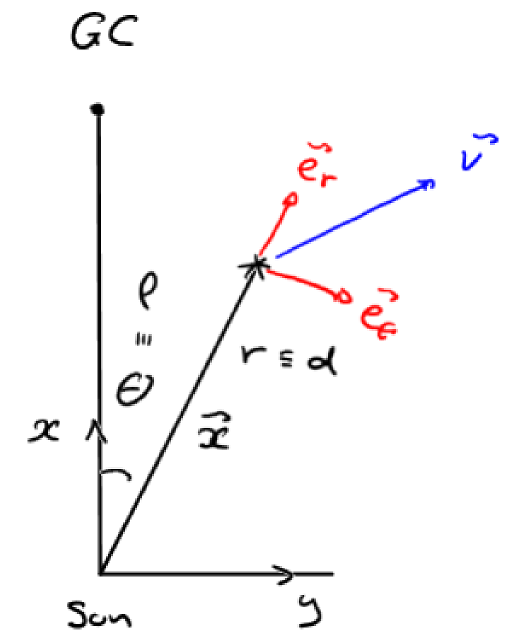
$$\begin{cases} A = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & K = \frac{1}{2}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) \\ B = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) & C = \frac{1}{2}\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right) \end{cases}$$

<u>Note</u>: we could do all this in 3D


K+C
A-B
A+B
K-C
10
5
0
−5
−10
−10
0
10

Motions of nearby stars

$\vec{V}_o = \vec{V}_\odot$



$\vec{x}$: position of a nearby star

$r \equiv d$: distance to the star

$$\vec{x} = \begin{cases} x = \underbrace{d \cos \ell}_{\text{galactocentric coord}} \equiv \underbrace{r \cos \theta}_{\text{polar coord}} \\ y = d \sin \ell \equiv r \sin \theta \end{cases}$$

Radial and tangential velocities

$$\begin{cases} V_r = \vec{v} \cdot \vec{e}_r = \vec{v} \cdot \dfrac{\vec{x}}{r} = \dfrac{1}{r} \left(x V_x + y V_y \right) \\ \quad = \cos \theta \, V_x + \sin \theta \, V_y \\ V_\theta = \left| \vec{v} \times \dfrac{\vec{x}}{r} \right| = \dfrac{1}{r} \left(x V_y - y V_x \right) \\ \quad = \cos \theta \, V_y - \sin \theta \, V_x \end{cases}$$

Re-write the Jacobian matrix in galacto-centric coordinates term of V_r and V_θ

$$\frac{\partial V_r}{\partial r} = \frac{\partial V_r}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V_r}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial V_r}{\partial x}\cos\theta + \frac{\partial V_r}{\partial y}\sin\theta$$

$$\frac{\partial V_r}{\partial \theta} = \dots \qquad \frac{\partial V_t}{\partial r} = \dots \qquad \frac{\partial V_t}{\partial \theta} = \dots$$

$$A = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$B = \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$C = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r} + \frac{\partial V_r}{\partial r}\right)$$

$$K = \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} + \frac{\partial V_r}{\partial r}\right)$$

See Shandrasekhar

$$A = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$B = \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$C = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r} + \frac{\partial V_r}{\partial r}\right)$$

$$K = \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} + \frac{\partial V_r}{\partial r}\right)$$

Purely axisymmetric disk

(no radial motions)

$$\frac{\partial}{\partial \theta} = 0 \qquad V_r = 0 \qquad \frac{\partial V_r}{\partial r} = 0$$

$$A = \frac{1}{2}\left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$B = \frac{1}{2}\left(-\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$C = 0$$

$$K = 0$$

With $V_\Theta = V_c$ (circular velocity)

$$A(R) := \frac{1}{2}\left[\frac{V_c(R)}{R} - \frac{d}{dR}V_c(R)\right] = -\frac{1}{2} R \frac{d\Omega(R)}{dR}$$

$$B(R) := -\frac{1}{2}\left[\frac{V_c(R)}{R} + \frac{d}{dR}V_c(R)\right] = -\left(\Omega(R) + \frac{1}{2} R \frac{d\Omega(R)}{dR}\right)$$

We can express Ω and κ from the Oort constants

$$\Omega = A - B$$

$$\kappa^2 = -4B(A-B) = -4B\Omega$$

Expressions of κ and ν from the total potential

$$\kappa^2(R_g) = \left.\frac{\partial^2\phi_{eff}}{\partial R^2}\right)_{(R_g,0)} = \left.\frac{\partial^2\phi}{\partial R^2}\right)_{(R_g,0)} + 3\frac{L_z^2}{R_g^4}$$

$L_z^2 = V_c^2 R_g^2$

$= R_g^3 \left.\frac{\partial\phi}{\partial R}\right)_{R_g}$

$$= \left.\frac{\partial^2\phi}{\partial R^2}\right)_{(R_g,0)} + \frac{3}{R_g}\left.\frac{\partial\phi}{\partial R}\right)_{(R_g,0)}$$

circ. frequency $\Omega^2 = \frac{1}{R_g}\left.\frac{\partial\phi}{\partial R}\right)_{R_g}$

$$= \left.\frac{\partial^2\phi}{\partial R^2}\right)_{(R_g,0)} + 3\Omega^2$$

$\Omega^2 = \frac{V_c^2}{R^2}$

$$= \left(R\frac{\partial(\Omega^2)}{\partial R} + 4\Omega^2\right)_{(R_g,0)}$$

$$= \left(\frac{1}{R}\frac{\partial(V_c^2)}{\partial R} + 2\Omega^2\right)_{(R_g,0)} = \left(\frac{1}{R}\frac{\partial(V_c^2)}{\partial R} + 2\frac{V_c^2}{R^2}\right)_{(R_g,0)}$$

Rigid rotation $\Omega = \text{cte}$

$$\frac{\partial}{\partial \theta} = 0 \qquad v_r = 0 \qquad \frac{\partial V_r}{\partial r} = 0 \qquad \frac{\partial \Omega}{\partial r} = 0$$

$$A = \frac{1}{2}\left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$B = \frac{1}{2}\left(-\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$C = 0$$

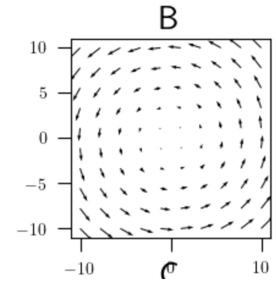
$$K = 0$$

$$A = 0 \qquad C = 0$$

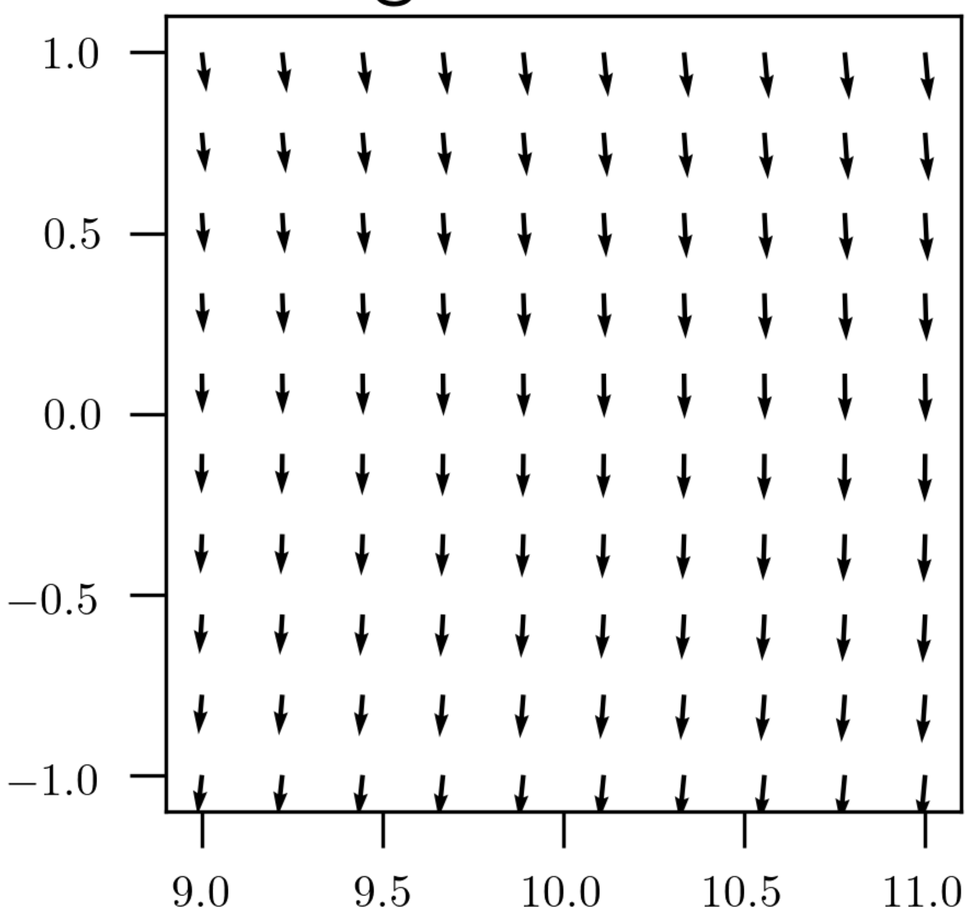
$$B = -\Omega \qquad K = 0$$

$$J = \begin{pmatrix} 0 & A-B \\ A+B & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$$

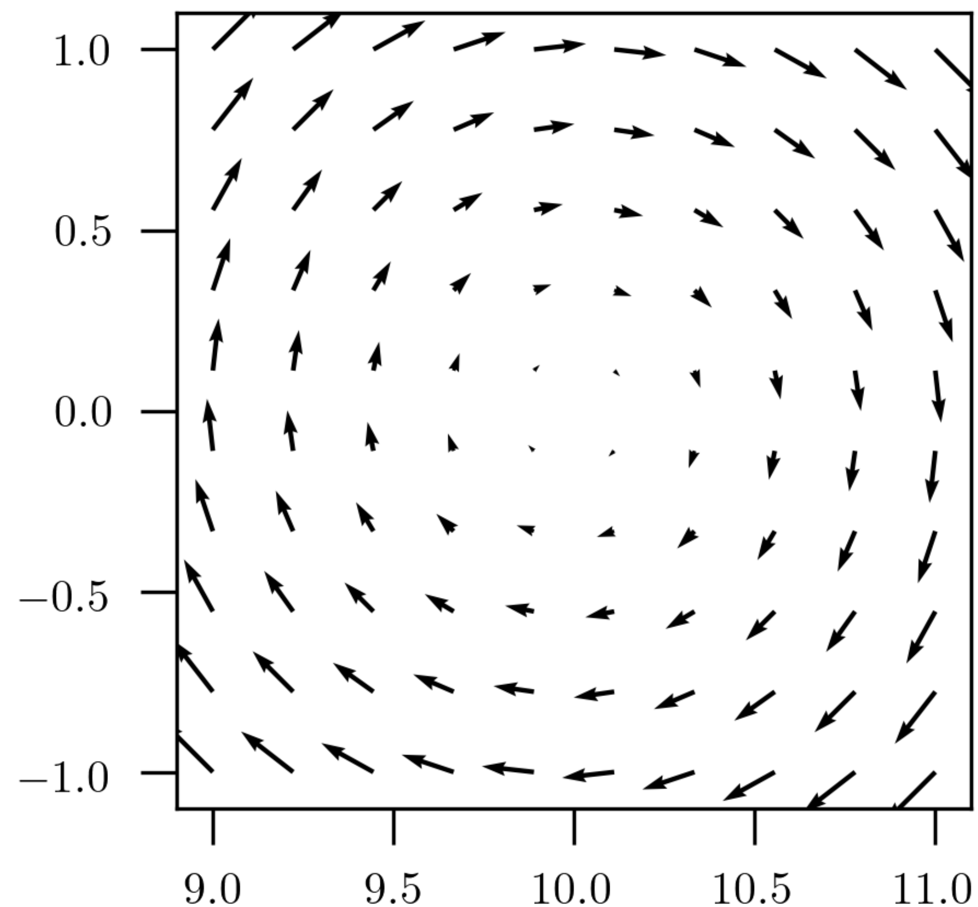
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



rigid rotation



differencial velocities



Rigid rotation $\Omega = \text{cte}$

$\frac{\partial}{\partial\theta} = 0 \quad v_r = 0 \quad \frac{\partial V_r}{\partial r} = 0 \quad \frac{\partial\Omega}{\partial r} = 0$

A	$=$	0	C	$=$	0
B	$=$	$-\Omega$	K	$=$	0

$$A = \frac{1}{2}\left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$B = \frac{1}{2}\left(-\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r}\right)$$

$$C = 0$$

$$K = 0$$

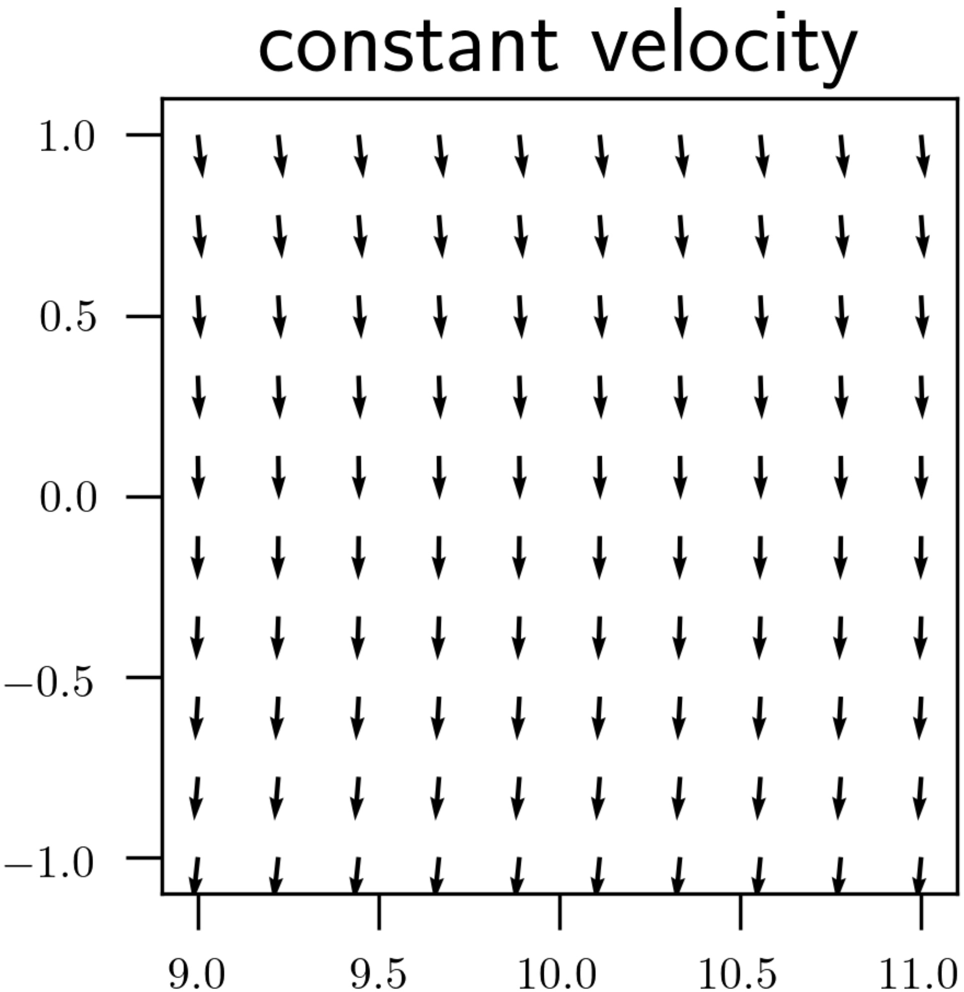
$$J = \begin{pmatrix} 0 & A-B \\ A+B & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$$

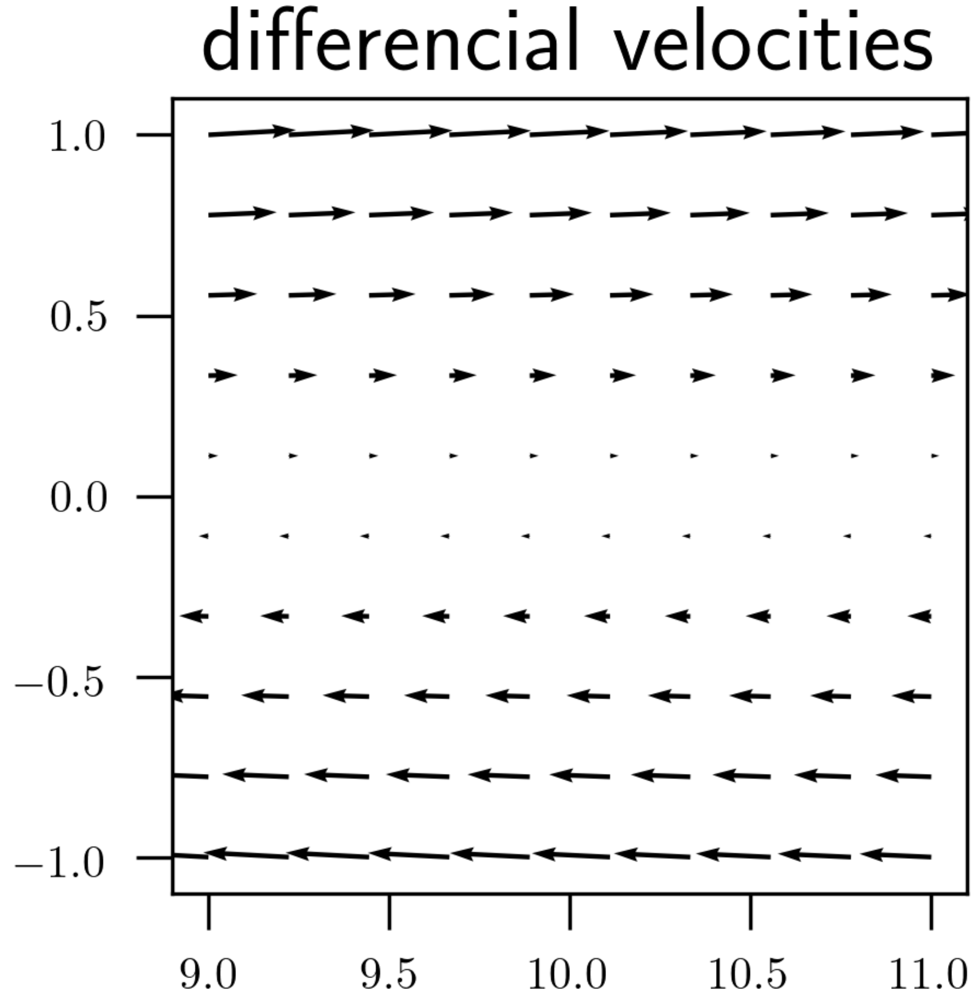
Constant rotation curve

$\frac{\partial}{\partial\theta} = 0 \quad v_r = 0 \quad \frac{\partial V_r}{\partial r} = 0 \quad \frac{\partial V_\theta}{\partial r} = 0$

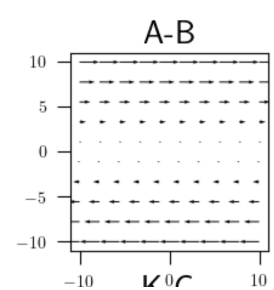
A	$=$	$\frac{1}{2}\Omega$	C	$=$	0
B	$=$	$-\frac{1}{2}\Omega$	K	$=$	0

$$J = \begin{pmatrix} 0 & A-B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ 0 & 0 \end{pmatrix}$$





$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



Keplerian decrease

$$\frac{\partial}{\partial \theta} = 0 \qquad v_r = 0 \qquad \frac{\partial v_r}{\partial r} = 0 \qquad v_\theta \sim r^{-1/2}$$

$$A = \frac{1}{2}\left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$B = \frac{1}{2}\left(-\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$C = 0$$

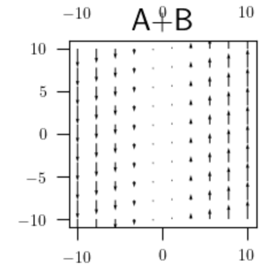
$$K = 0$$

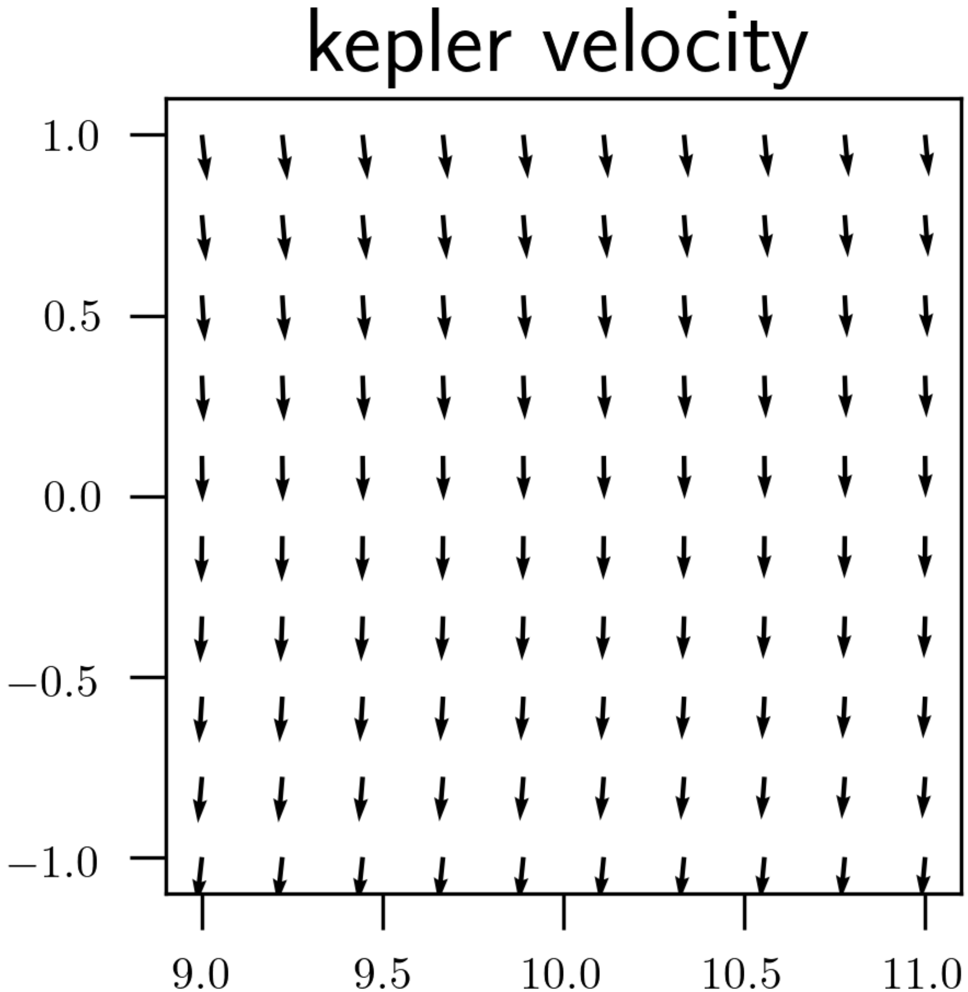
$$A = \frac{3}{4}\Omega \qquad C = 0$$

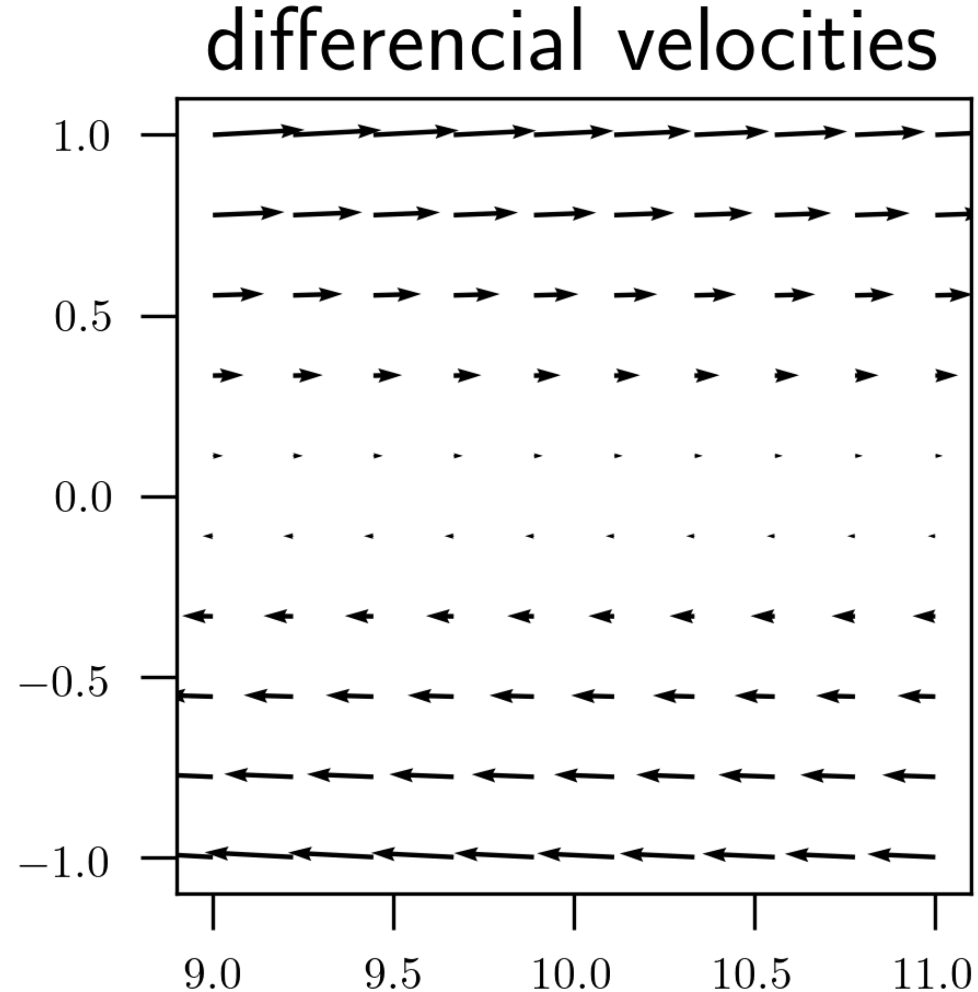
$$B = -\frac{1}{4}\Omega \qquad K = 0$$

$$J = \begin{pmatrix} 0 & A-B \\ A+B & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \Omega \\ \frac{1}{2}\Omega & 0 \end{pmatrix}$$

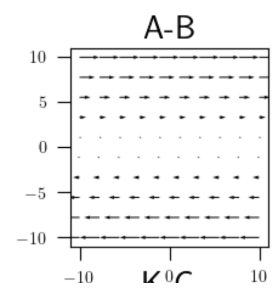
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



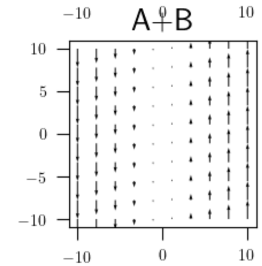




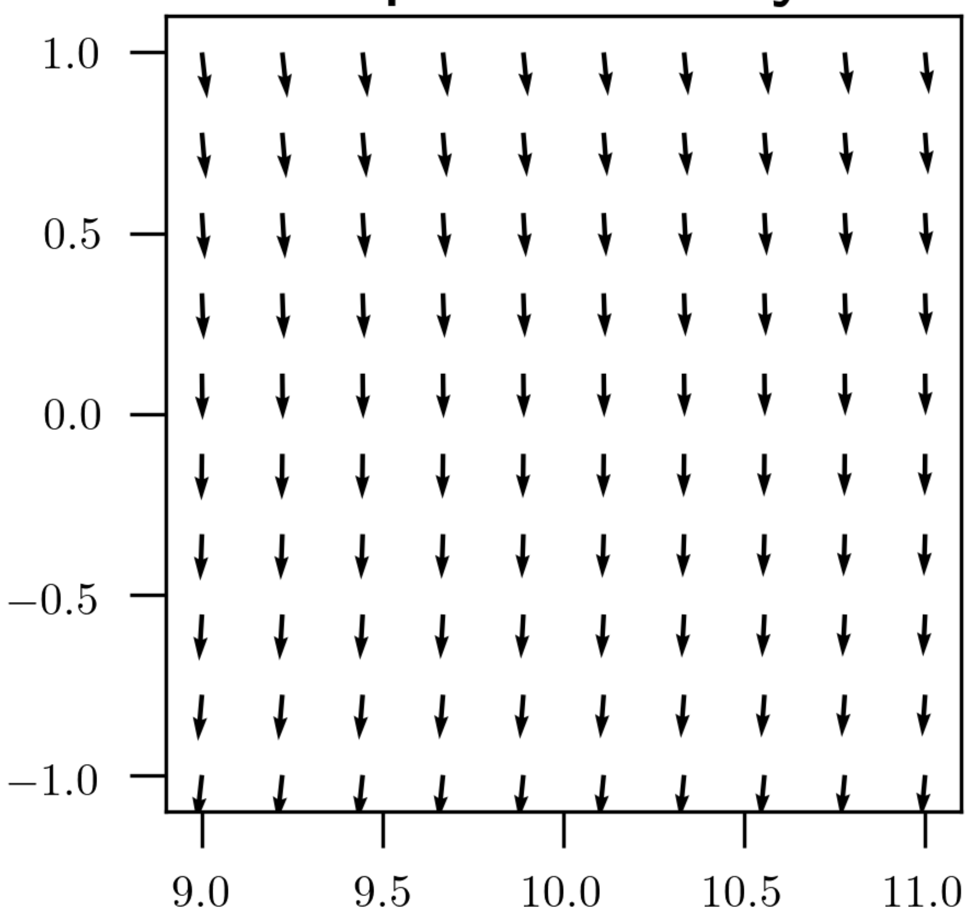
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



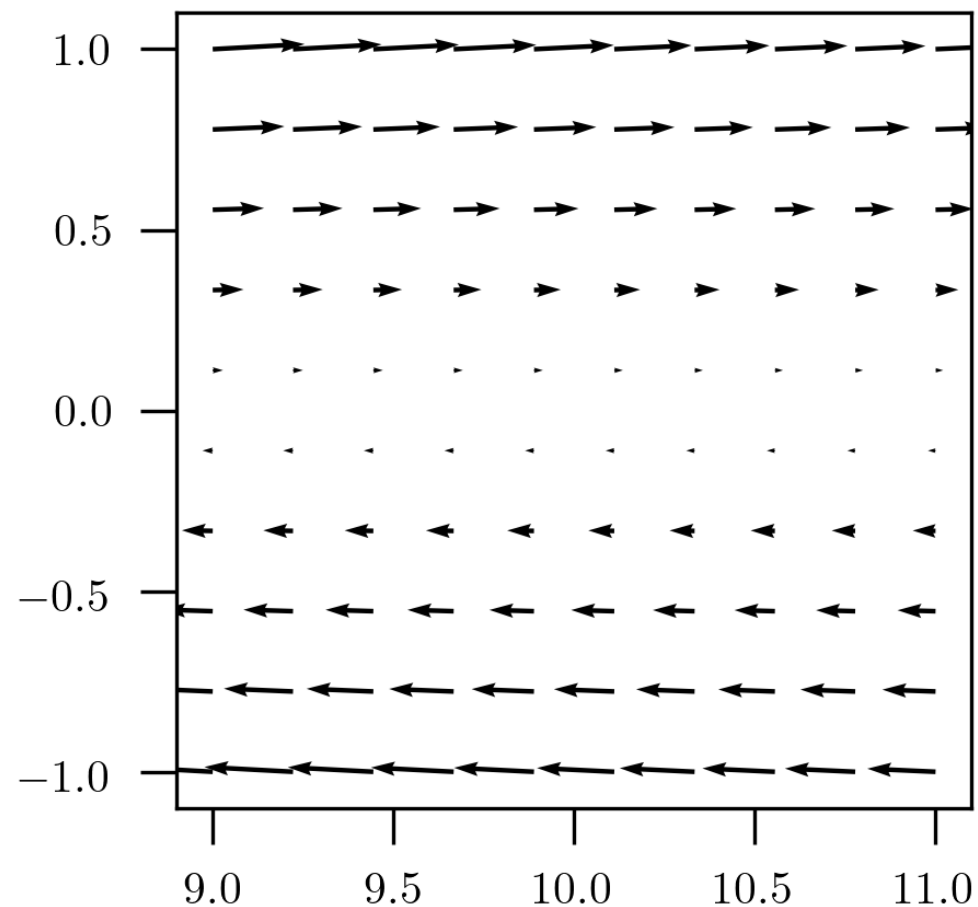
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



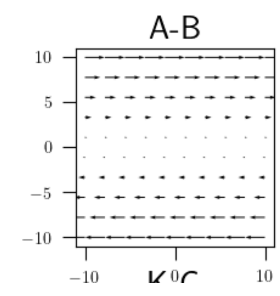
kepler velocity



differencial velocities



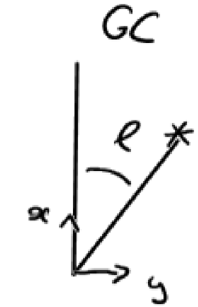
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



Can we measure the Oort constants (at the Sun location)?

In Galacto Centric coordinates

(ℓ, b, d), with $d = r$



with

$$\begin{cases} x = d\cos\ell \\ y = d\sin\ell \end{cases} \qquad \begin{cases} V_x = (K + C)\,x + (A - B)\,y \\ V_y = (A + B)\,x + (K - C)\,y \end{cases}$$

- $V_r = \vec{v} \cdot \dfrac{\vec{x}}{r} = \dfrac{1}{r}\left(x V_x + y V_y \right) \equiv V_x \cos\ell + V_y \sin\ell$

$$\boxed{V_r = d\left[K + C\cos(2\ell) + A\sin(2\ell) \right]}$$

- $V_t = \left| \vec{v} \times \dfrac{\vec{x}}{r} \right| = \dfrac{1}{r}\left(x V_y - y V_x \right) \equiv -V_x \sin\ell + V_y \cos\ell$

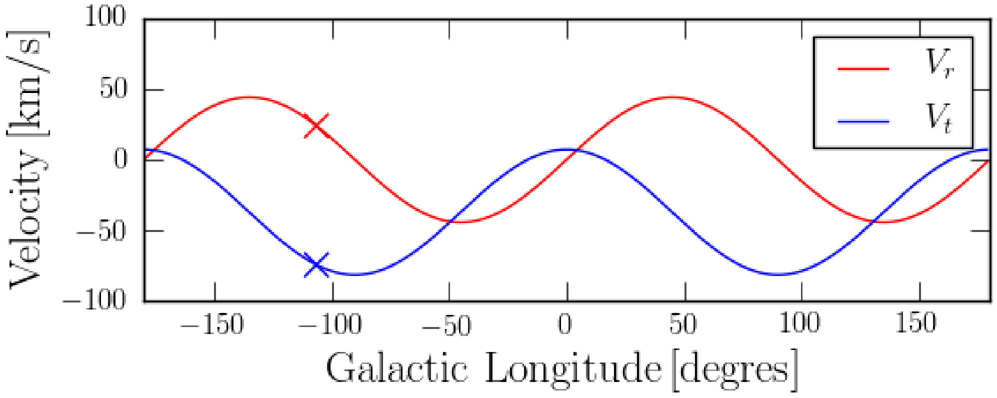
$$\boxed{V_t = d\left[B + A\cos(2\ell) - C\sin(2\ell) \right]}$$

In the axisymmetric case $C = K = 0$

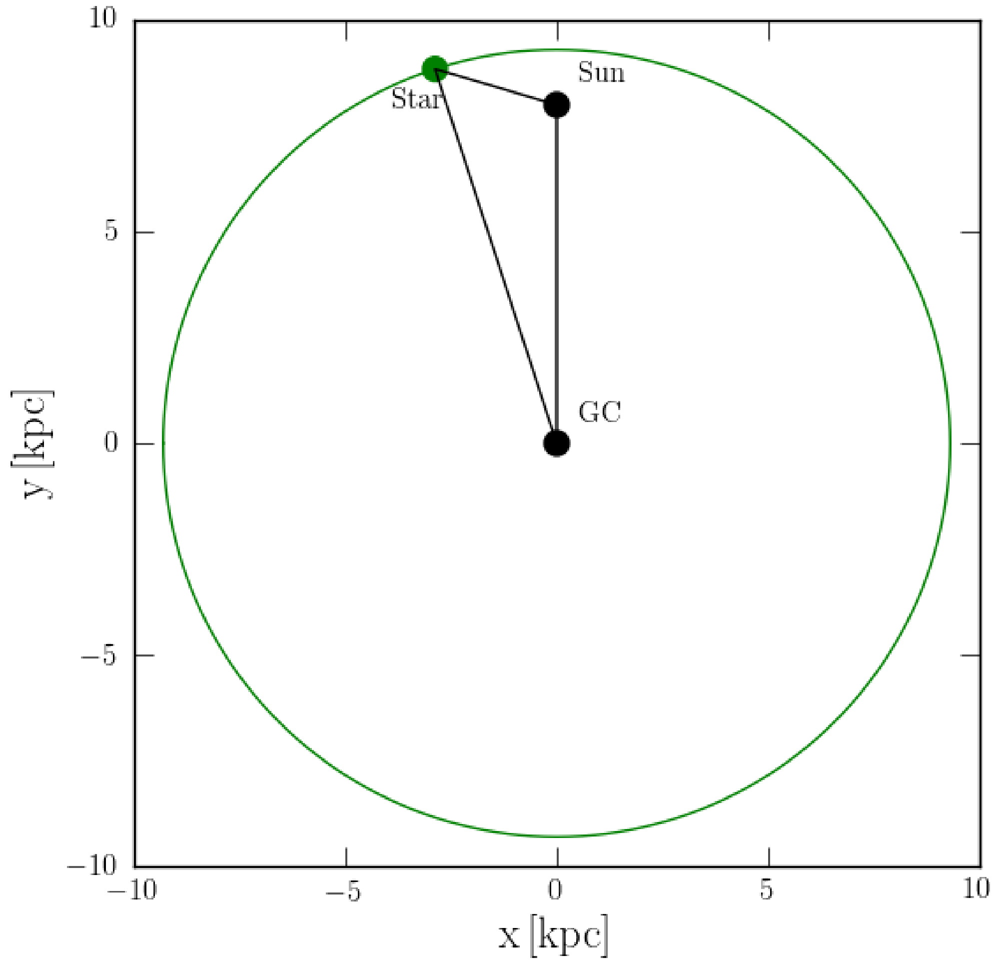
(purely circular orbits)

$$\begin{cases} v_r = A d \sin(2\ell) \\ v_t = A d \cos(2\ell) + B d \end{cases}$$

The Oort constants

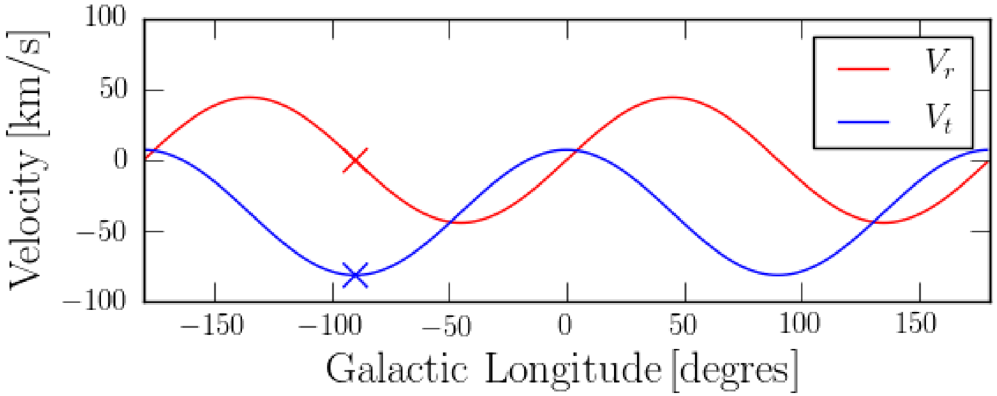


$$\begin{cases} V_r = Ad\sin(2l) \\ V_t = Ad\cos(2l) + Bd \end{cases}$$

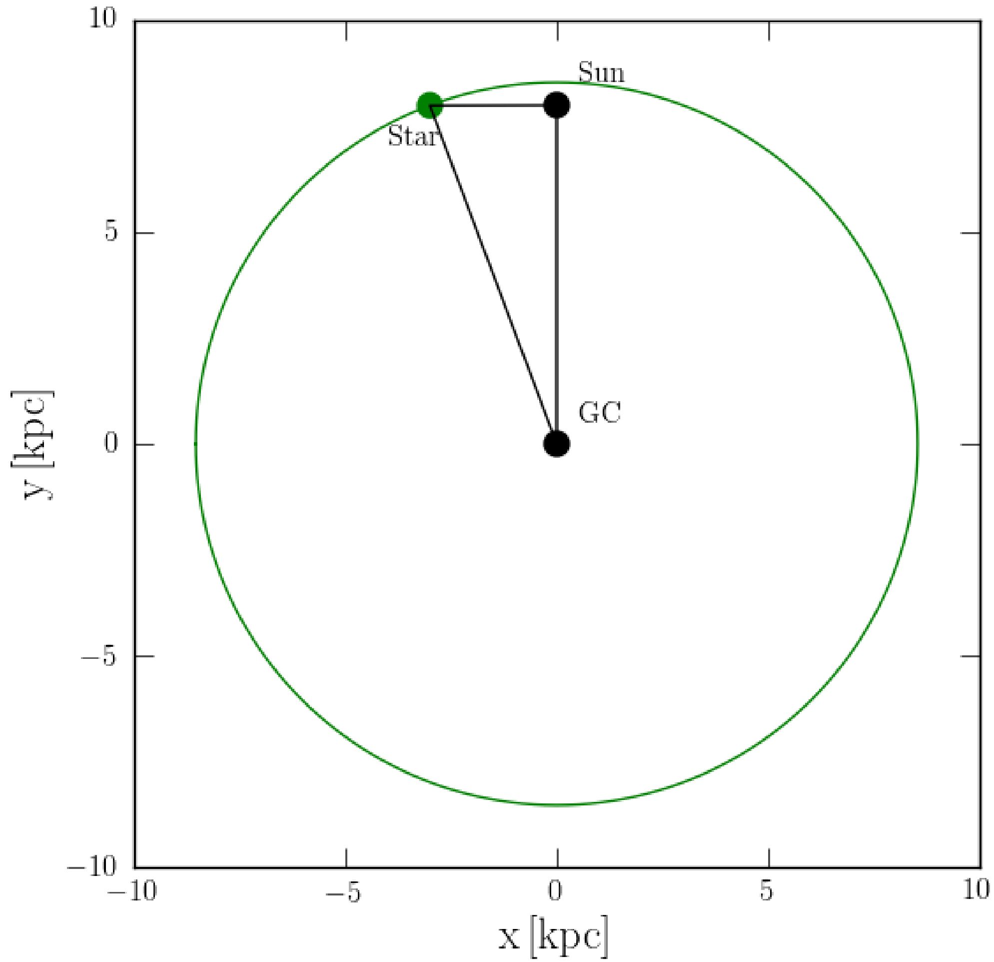


The Oort constants

$$l = -90$$



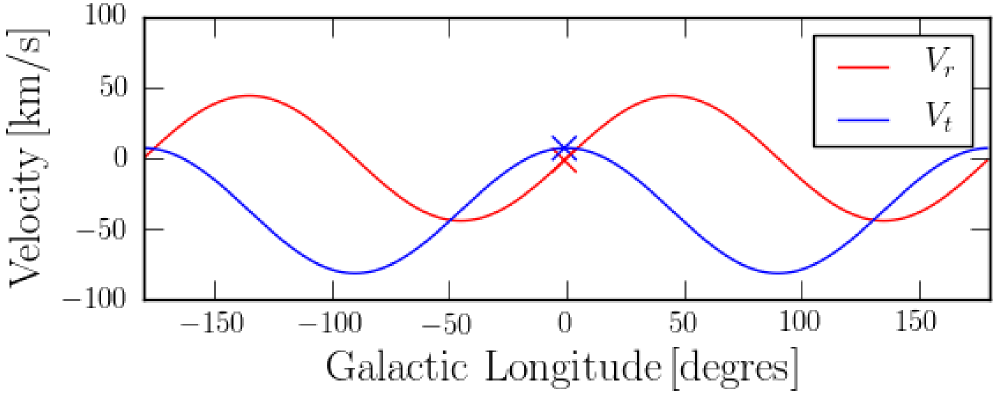
$$\begin{cases} V_r = Ad\sin(2l) \\ V_t = Ad\cos(2l) + Bd \end{cases}$$



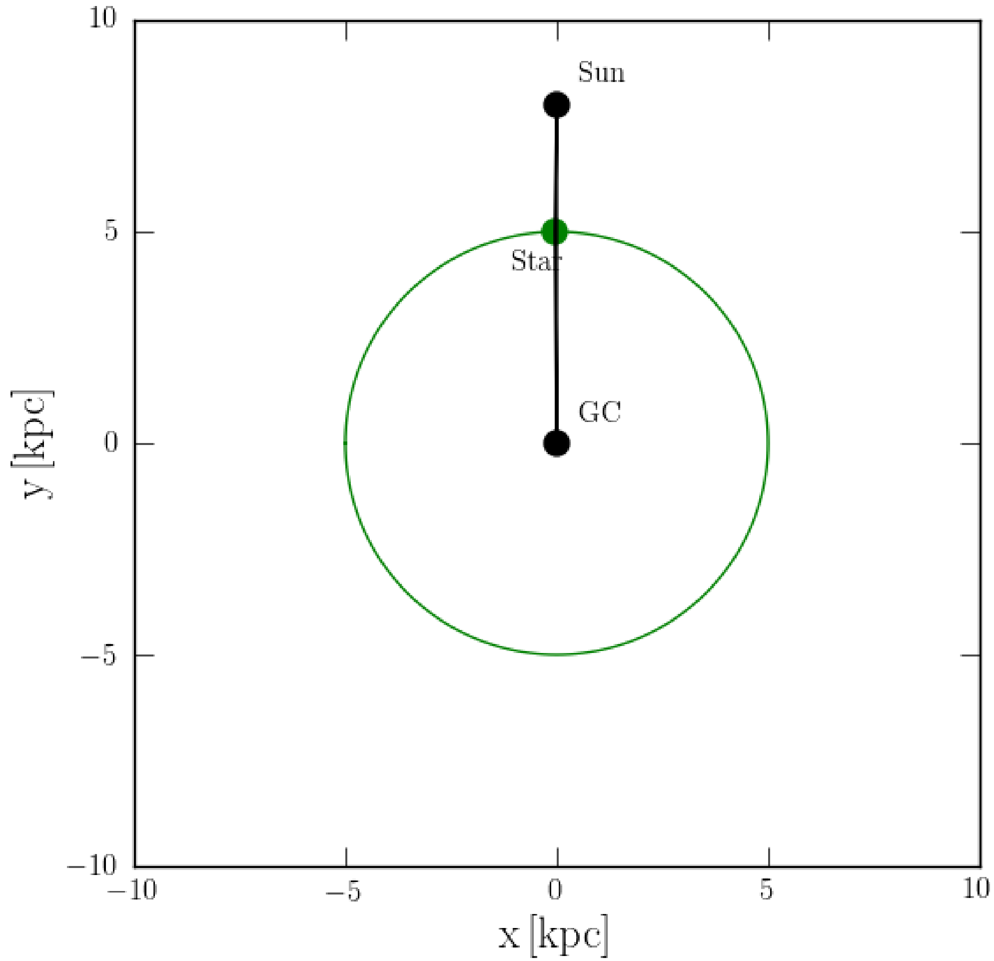
$$\begin{cases} V_r = 0 \\ V_t = (A + B)d \end{cases}$$

The Oort constants

$l = 0$



$$\begin{cases} V_r = Ad\sin(2l) \\ V_t = Ad\cos(2l) + Bd \end{cases}$$



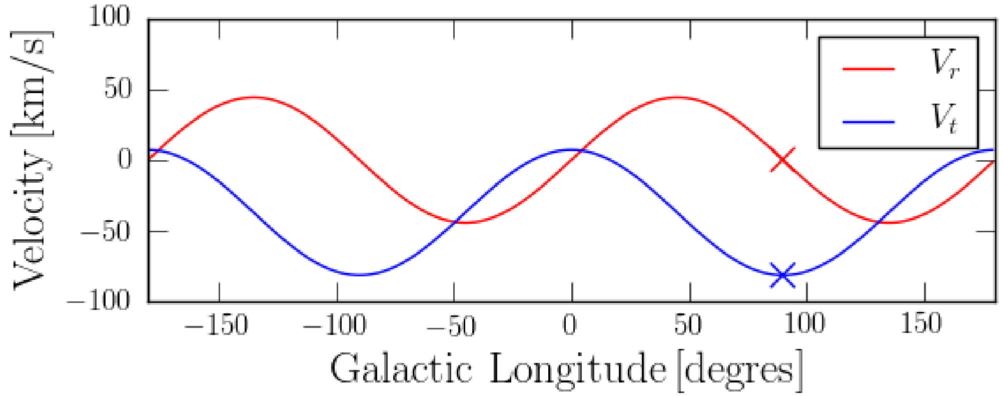
$$\begin{cases} V_r = 0 \\ V_t = (A + B)d \end{cases}$$

$\Omega = \text{cte}$

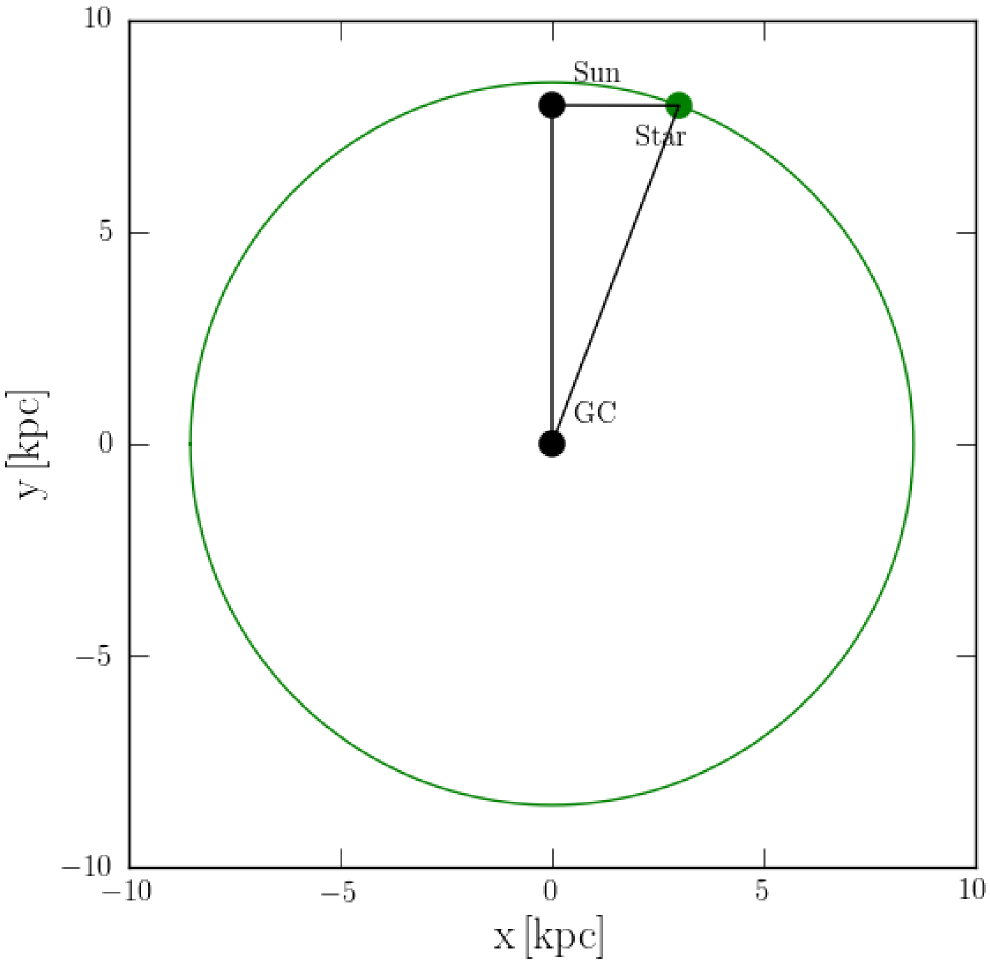
$$\begin{cases} V_r = 0 \\ V_t = -\Omega d \end{cases}$$

The Oort constants

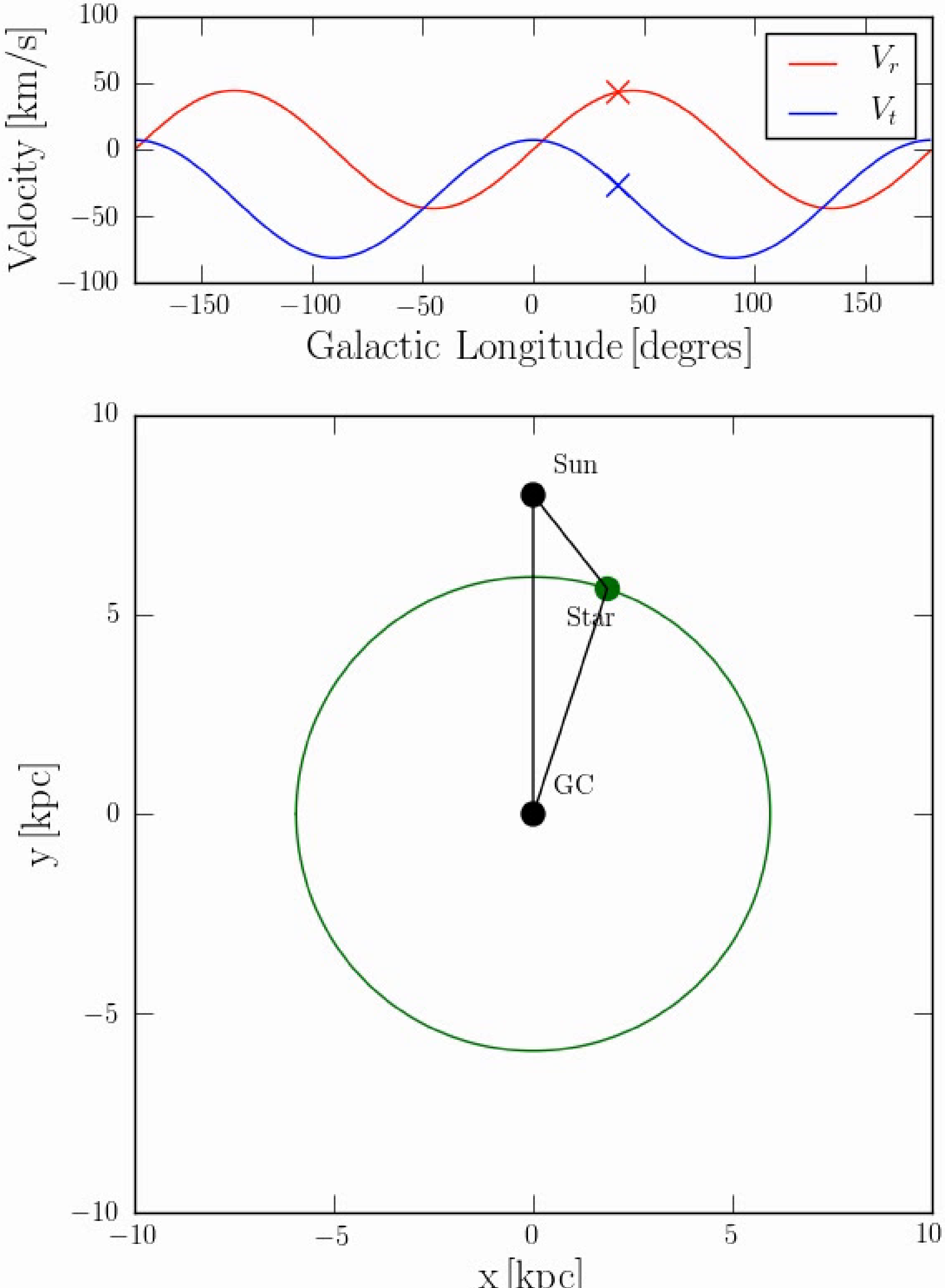
$$l = 90$$



$$\begin{cases} V_r = Ad\sin(2l) \\ V_t = Ad\cos(2l) + Bd \end{cases}$$

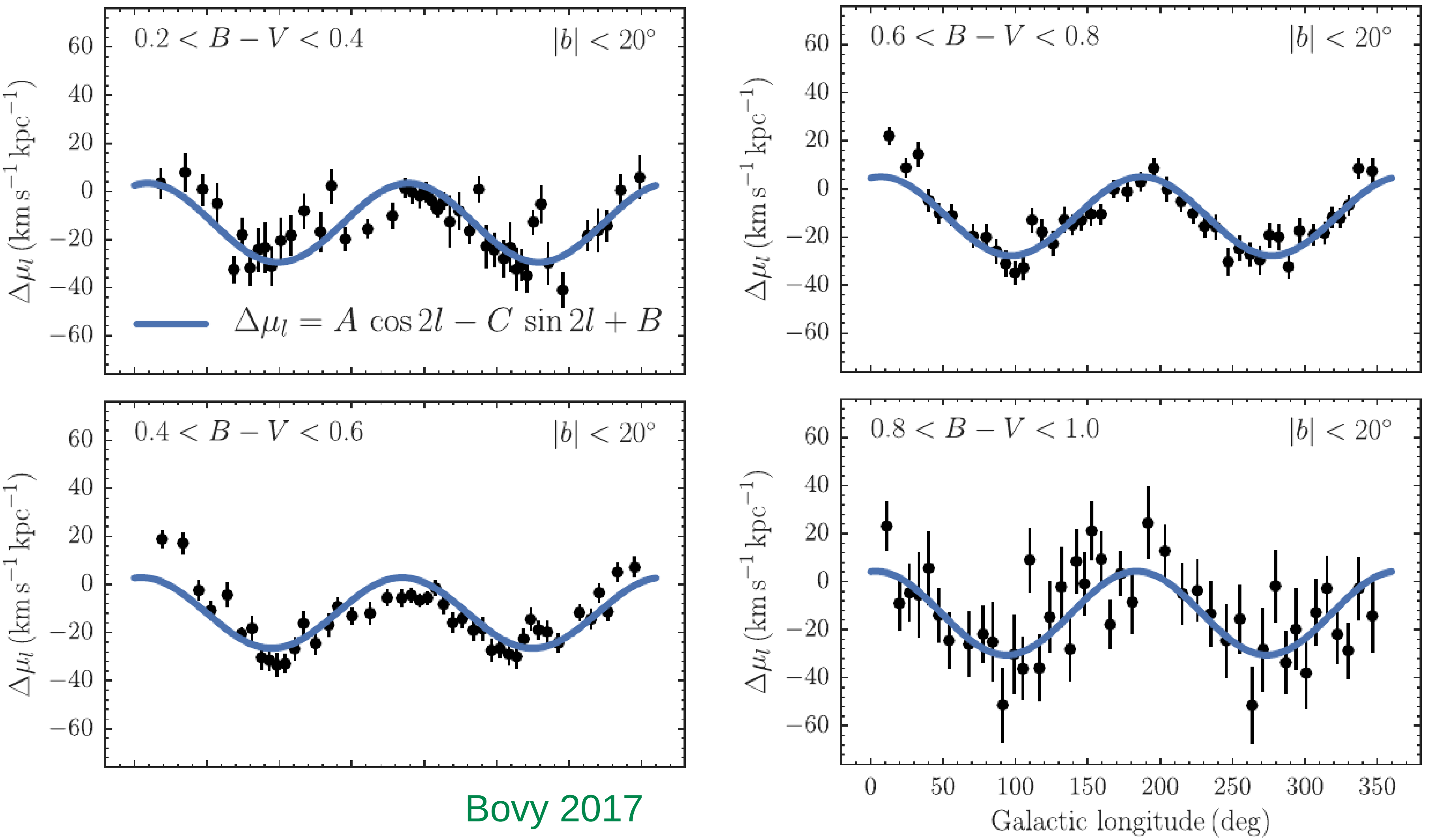


$$\begin{cases} V_r = 0 \\ V_t = (A+B)d \end{cases}$$



Velocity [km/s]
Galactic Longitude [degres]
V_r
V_t
Sun
Star
GC
y [kpc]
x [kpc]

Proper motions measurements with GAIA



Bovy 2017

Figure 2. Comparison between the observed mean proper motion in Galactic longitude corrected for the solar motion (see equation 3) as a function of l and the best-fitting model for the four main colour bins used in the analysis. The data clearly display the expected signatures due to the differential rotation of the Galactic disc. The agreement between the model and the data is good.

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Galactic rotation in *Gaia* DR1

The Oort constants

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$$A = 15.3 \pm 0.4\,\mathrm{km\,s^{-1}\,kpc^{-1}} \qquad B = -11.9 \pm 0.4\,\mathrm{km\,s^{-1}\,kpc^{-1}}$$

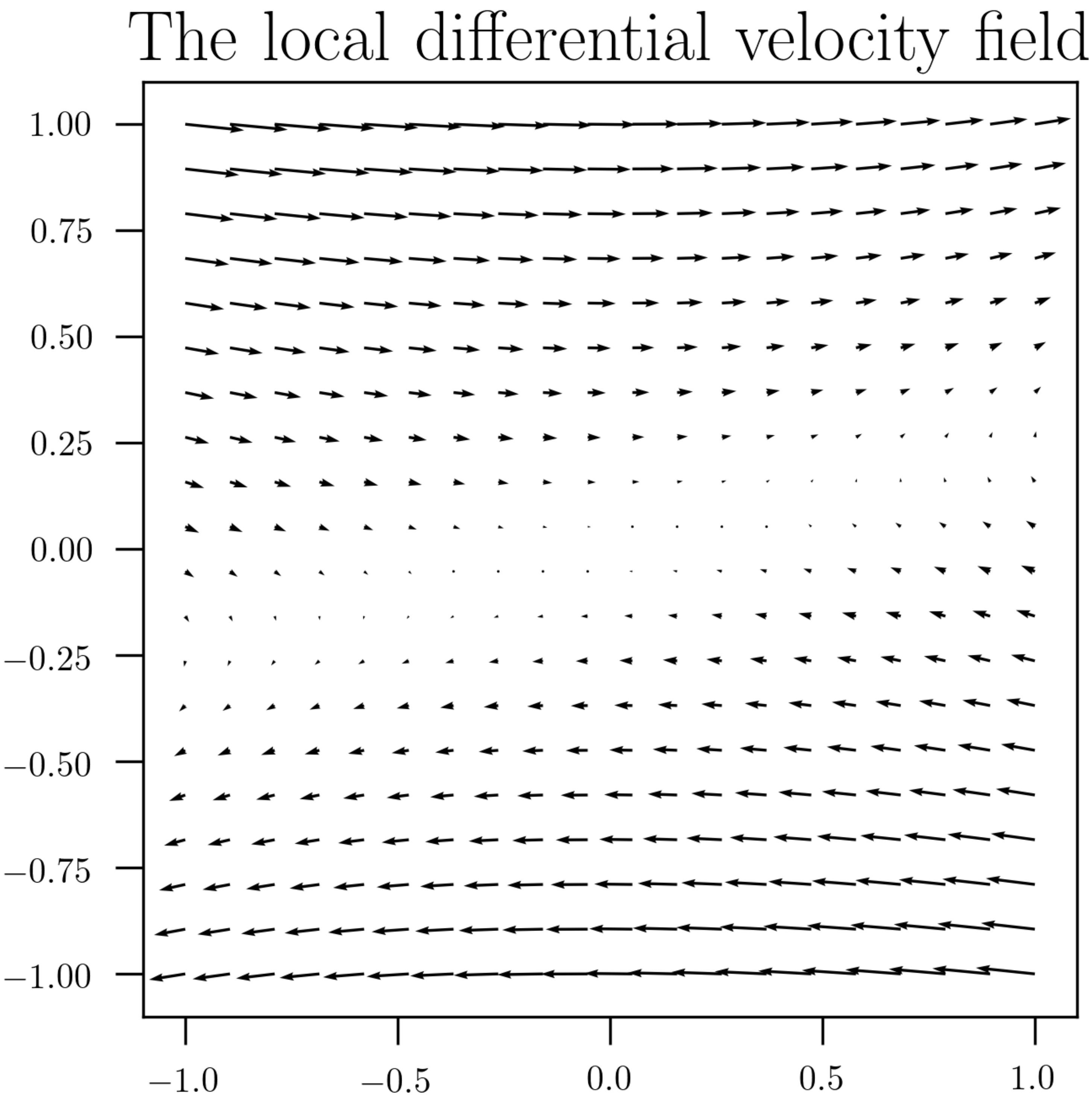
$$C = -3.2 \pm 0.4\,\mathrm{km\,s^{-1}\,kpc^{-1}} \qquad K = -3.3 \pm 0.6\,\mathrm{km\,s^{-1}\,kpc^{-1}}$$

using
$$\begin{cases} \Omega = A - B \\ \kappa^2 = -4B\,(A - B) = -4\,B\,\Omega \end{cases}$$

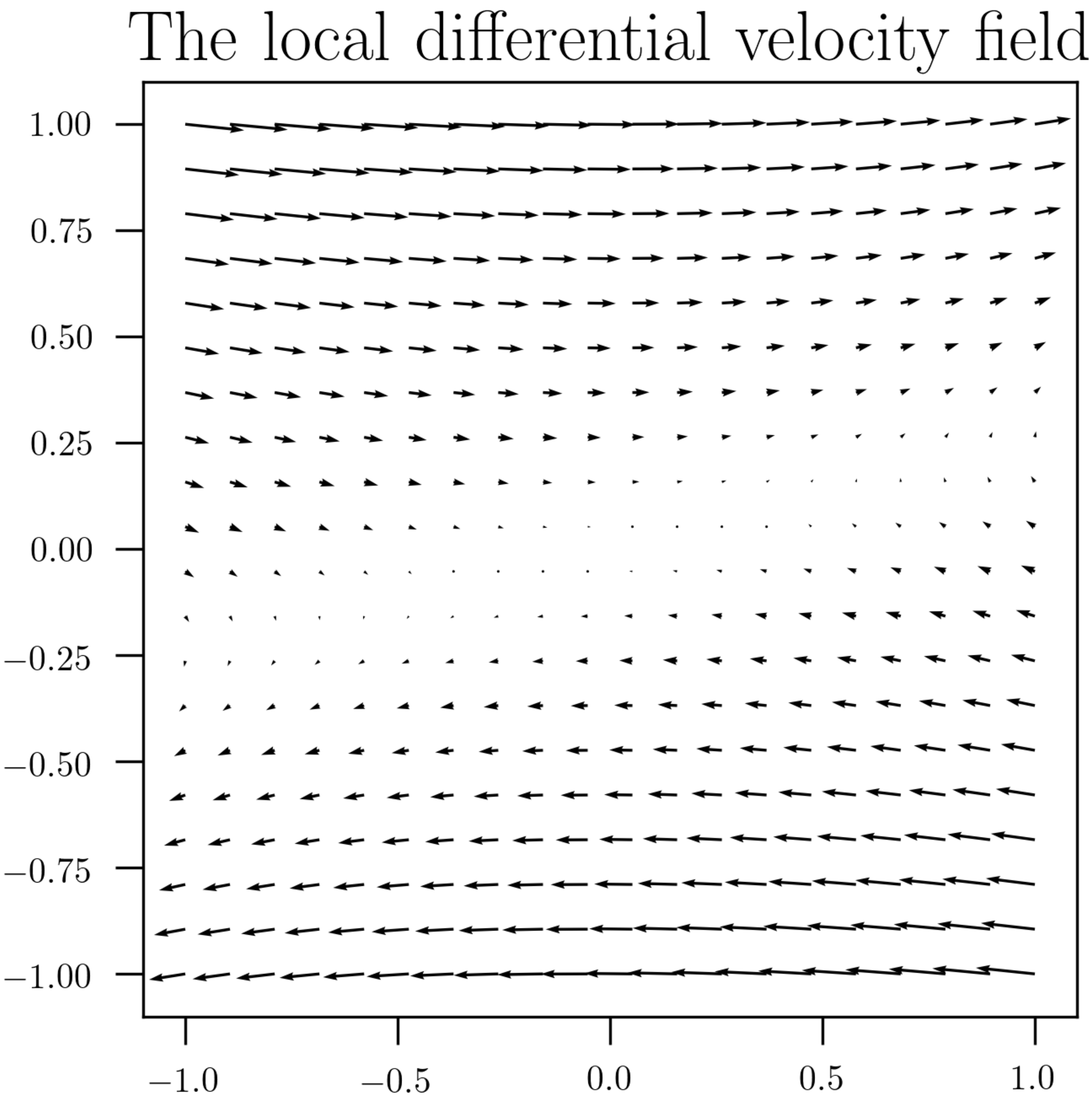
$$\begin{cases} \Omega_0 = 27 \pm 0.8\,\mathrm{km\,s^{-1}\,kpc^{-1}} \\ \kappa_0 = 35 \pm 0.2\,\mathrm{km\,s^{-1}\,kpc^{-1}} \end{cases} \qquad \begin{cases} T_\phi \cong 227\,\mathrm{Myr} \\ T_R \cong 175\,\mathrm{Myr} \end{cases}$$

$$\frac{\kappa_0}{\Omega_0} = 2\,\frac{-B}{A - B} \cong 1.29 \qquad \kappa_0 > \Omega_0 \quad \text{as expected}$$

Shear (A-B) + compression (K+C)



Shear (A-B) + compression (K+C)



Stellar Orbits

$$\nu$$

and the density relation

Galaxy properties from the vertical frequency ν

What can we learn from κ, Ω, ν ratios ?

Poisson equation in cylindrical coordinates

$$\nabla^2 \phi(R,z) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right) + \frac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho(R,z)$$

$z = 0$

$$\nabla^2 \phi(R,z) = \frac{1}{R}\frac{\partial}{\partial R}\left(V_c^2\right) + \nu^2 = 4\pi G \rho(R,z=0)$$

① if $\rho(R,z) \rightarrow \delta(z)\Sigma(R)$ $\qquad \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right) \ll \frac{\partial^2 \phi}{\partial z^2}$

② if $V_c = cte$, $\qquad \frac{1}{R}\frac{\partial}{\partial R}\left(V_c^2\right) = 0$

so,

$$\nu^2 = 4\pi G \rho(R,z=0) \qquad ①$$

Expected relation between $\varkappa$ and ν ?

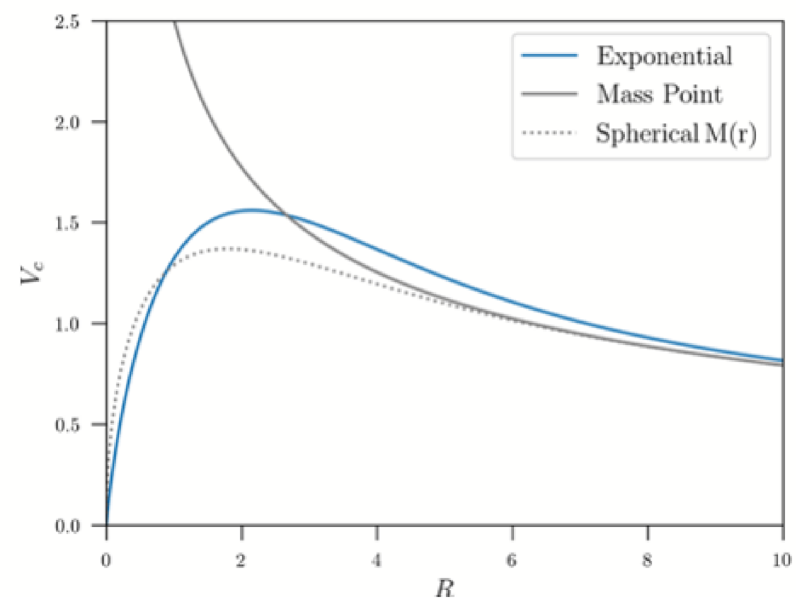
In spherical systems : $\Omega^2 = \frac{GM(r)}{r^3} = \frac{4}{3}\pi G\bar{\rho}$ $\qquad \bar{\rho} = \frac{M(r)}{\frac{4}{3}\pi r^3}$

As V_c for a cylindrical distribution is not so different than a spherical one if $M(r) = M(R)$

and $\Omega \sim \frac{1}{V_c}$

For an axisymmetric disk, we can estimate

$$\Omega^2 = \frac{GM(R)}{R^3} = \frac{4}{3}\pi G\bar{\rho} \qquad (2)$$



For the flat rotation curve part, we have

$$\varkappa^2 = \frac{1}{R} \frac{\partial}{\partial R}(V_c^2) + 2\Omega^2 \quad \Rightarrow \quad \boxed{\varkappa^2 = 2\Omega^2} \quad (3)$$

Combining (1) + (2) + (3)

$$\boxed{\frac{\nu^2}{\varkappa^2} = \frac{3}{2} \frac{\rho}{\bar{\rho}}}$$

← density in the plane

← mean density computed inside the radius

Estimation of the vertical frequency

From

$$\frac{\nu^2}{\kappa^2} = \frac{3}{2}\frac{\rho_{\rm d}}{\bar{\rho}}$$

$$\frac{4}{3}\pi G\bar{\rho} = \Omega_{\odot}^2 = \frac{V_{\rm c,\odot}^2}{R_{\odot}^2} \qquad \begin{cases} V_{\rm c,\odot} \cong 200\,{\rm km/s} \\ R_{\odot}^2 \cong 8\,{\rm kpc} \end{cases}$$

$$\Rightarrow \bar{\rho} \cong 0.039\frac{\rm M_{\odot}}{\rm pc^3}$$

and with

$$\rho_{\rm d} \cong 0.1\frac{\rm M_{\odot}}{\rm pc^3}$$

$$\begin{cases} T_{\phi} \cong 227\,{\rm Myr} \\ T_R \cong 175\,{\rm Myr} \end{cases}$$

$$\frac{\nu}{\kappa} \cong 2$$

$$T_z = \frac{T_R}{2} \cong 87\,{\rm Myr}$$

Stellar Orbits

Integral of motion and Surfaces of section

Integrals of motion

A stellar orbit defines a path in the 6-D phase space ($x, y, z, \dot{x}, \dot{y}, \dot{z}$ in cartesian coordinates)

Definition :

An integral of motion $I[\mathbf{x}, \mathbf{v}]$ is any function of the phase-space co-ordinates alone that is constant along an orbit:

$$I[\mathbf{x}(t_1), \mathbf{v}(t_1)] = I[\mathbf{x}(t_2), \mathbf{v}(t_2)]$$

Examples :

- Hamiltonian $H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = E$
- Total angular momentum $\vec{L}: \; L_x = \mathrm{cte}, \; \mathrm{L_y} = \mathrm{cte}, \; \mathrm{L_z} = \mathrm{cte}$
- z-component of the angular momentum $L_z = \mathrm{cte}$

Remarks :

- Orbits may have between 0 to 5 integrals of motion.
- Integrals of motion may exist without an analytical form.

Integrals of motion

Interest of integrals of motion :

Restrict the study of an orbit to a subset of the phase space

Example I :

Orbit in spherical potentials

- 6-D

6 indep. variables $(x, y, z, \dot{x}, \dot{y}, \dot{z})$

- Angular momentum conservation
  3 integrals, 2 among the three

$\vec{n} = \vec{L}/|\vec{L}|$ defines a plane → 4-D

4 indep. variables $(r, \phi, \dot{r}, \dot{\phi})$

- Angular momentum conservation + energy

2 indep. variables (r, ϕ)

$L = |\vec{L}| \quad E$ → 2-D

$$\dot{\phi} = \frac{L}{r^2}$$

$$\dot{r} = \pm\sqrt{2(E - \Phi(r)) - L^2/r^2}$$

defines a 2-D surface in the phase space

Given $E, \ \vec{L}$ the position and velocities of a star (i.e. the position in the phase space) is fully determined by providing two additional quantities, ex: $r, \ \phi$

Integrals of motion

Is there a fifth integral ?

Example of the Keplerian potential :

We showed that:

$$r(\phi) = \frac{1}{C\,\cos(\phi-\phi_0) + \frac{GM}{L^2}}$$

with:

$$E = \frac{1}{2}\,(C\,L)^2 - \frac{1}{2}\left(\frac{GM}{L}\right)^2$$

we have then the new integral of motion:

$$\phi_0(r,\phi) = \phi - \arccos\left[\frac{1}{C(L,E)}\left(\frac{1}{r} - \frac{GM}{L^2}\right)\right]$$

→ 1-D
a curve

1 indep. variable (r)

Given $E,\ \vec{L},\ \phi_0$ the position and velocities of a star is fully determined by providing only one additional quantities, ex: r

Integrals of motion

Example II :

Orbit in axi-symmetric potentials

- 6-D

  6 indep. variables $(x, y, z, \dot{x}, \dot{y}, \dot{z})$

- z-component angular momentum conservation
  1 integral $\dot{\theta} = \frac{L_z}{R^2}$

  5 indep. variables $(R, z, \dot{R}, \dot{z}, \theta)$

- Initial azimuth $\theta(t) = L_z \int_{t_0}^{t} \frac{1}{R^2(t')}\, dt' + \theta_0$

  4 indep. variables $(R, z, \dot{R}, \dot{z})$
  4-D (meridional plane)

- Energy E

  not an integral, a constant !

  3 indep. variables $(R, z, \dot{R})$
  3-D

Given $E,\ L_z,\ \theta_0$ and t — the position and velocities of a star (in the phase space) is fully determined by providing three additional quantities, ex: $R, z, \dot{R}$

Given E — the position and velocities of a star (in the phase space of the meridional plane) is fully determined by providing three additional quantities, ex: $R, z, \dot{R}$

Is there a third integral ?

Surfaces of section

Can we visualize the phase phase and check if an additional integral of motion exists ?

Idea :

We study the orbits in the meridional plane

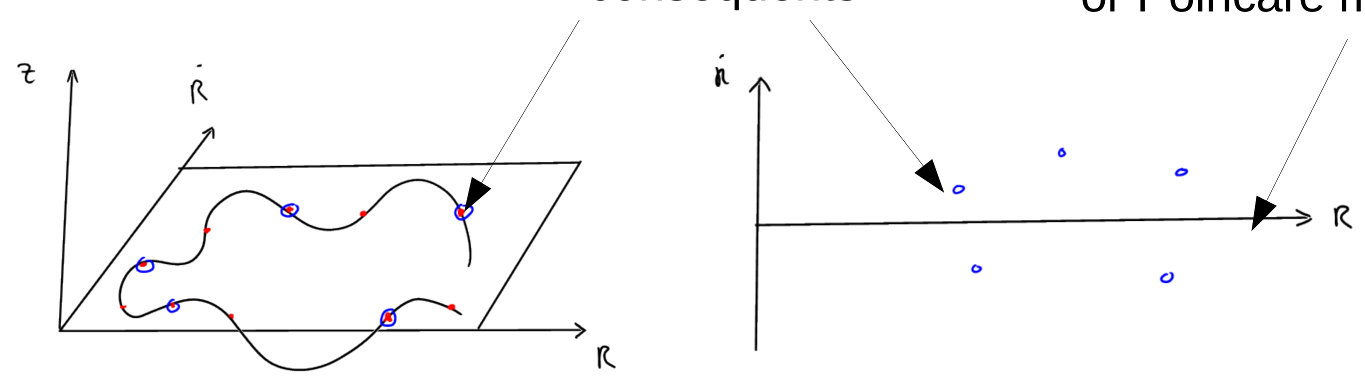
- 4-D — 4 indep. variables $(R, z, \dot{R}, \dot{z})$
- Energy E

→ 3-D — 3 indep. variables $(R, z, \dot{R})$

- Drawing a 3-D phase space is still not easy. Instead, we draw slices of the phase space. We plot only phase space points that:
  - cross the $z = 0$ plane
  - have $\dot{z} > 0$

consequents

Surface of section or Poincaré maps



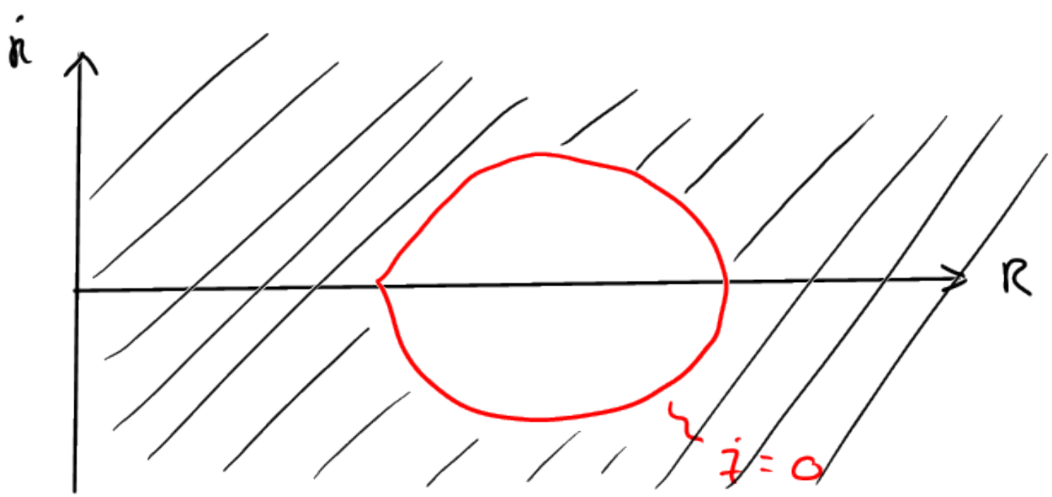
Surfaces of section

- A point in the surface of section (for a given E and L_z) defines an orbit as the three independent variables $(R, \dot{R}, z = 0)$ are defined.
- Even if orbits have the same energy, they will never intersect in the plane (EoM are first order diff. equations).
- Zero velocity curve : curve defined by $\dot{z} = 0$

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{\rm eff}(R, z = 0) \qquad \Rightarrow \qquad \dot{R} \leqslant \pm\sqrt{2\left[E - \Phi_{\rm eff}(R, z = 0)\right]}$$

$$\dot{R}(R) = \pm\sqrt{2\left[E - \Phi_{\rm eff}(R, z = 0)\right]}$$

defines the accessible region of the phase space

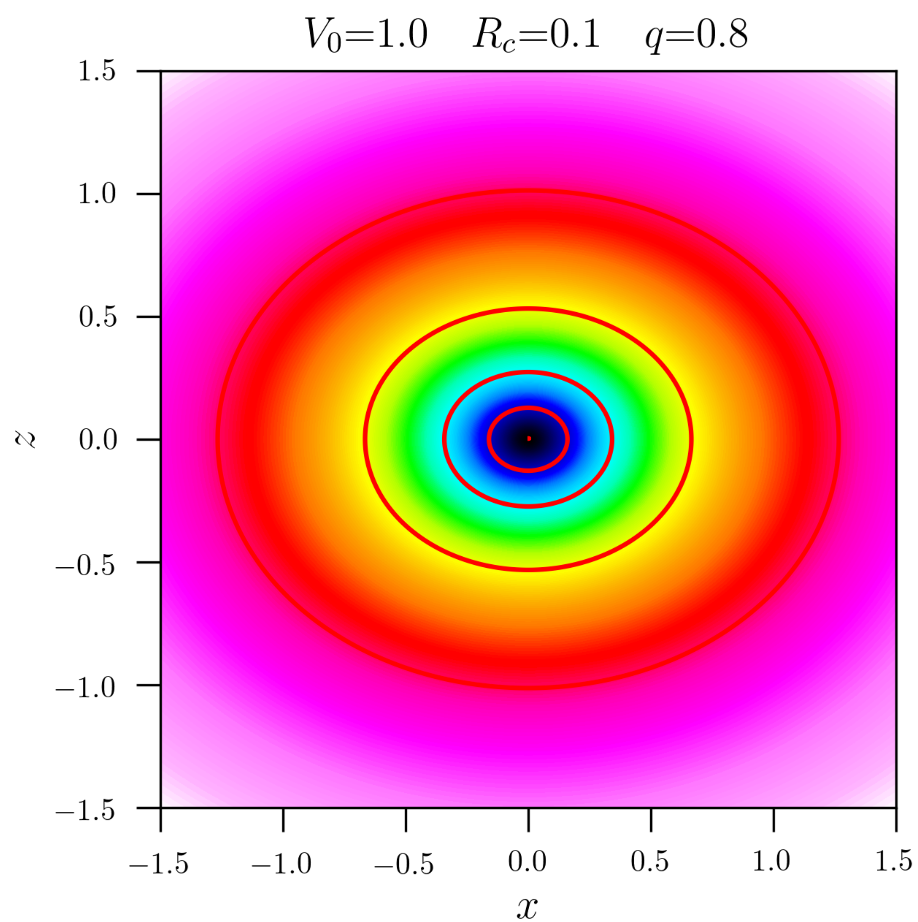


Surfaces of section

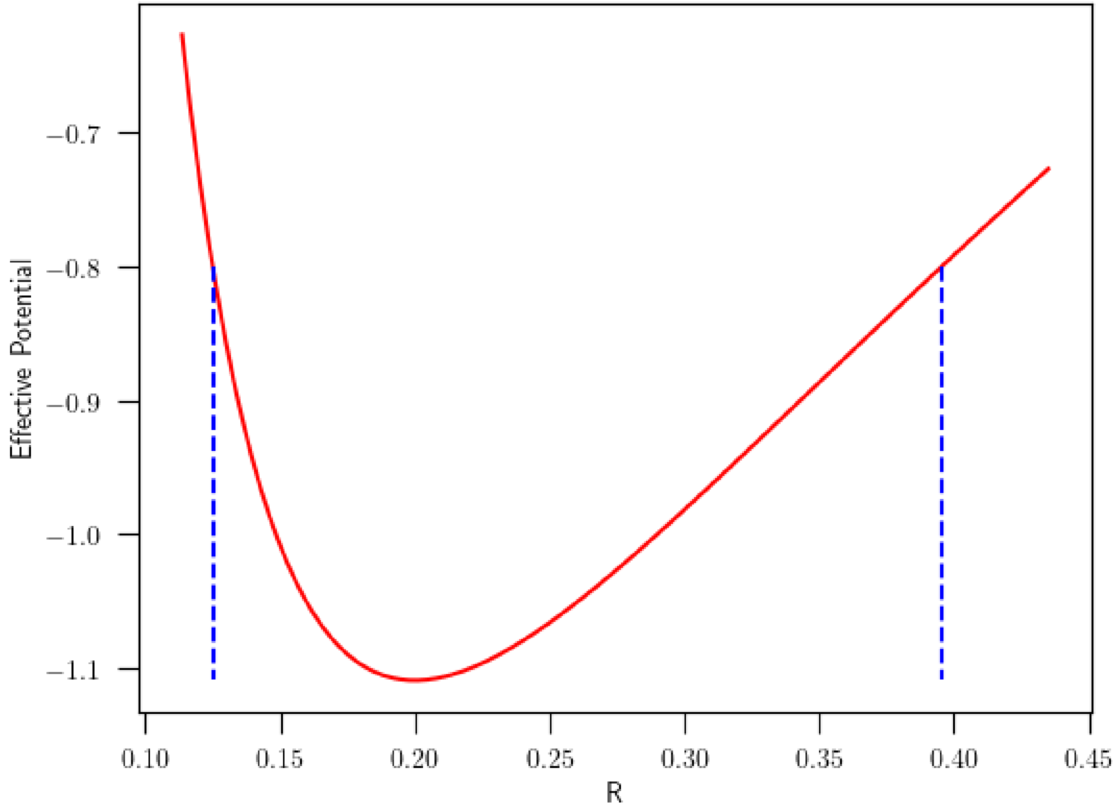
Examples

Logarithmic potential

$$\Phi_{\log}(R, z) = \frac{1}{2} V_0^2 \ln \left(R_c^2 + R^2 + \frac{z^2}{q^2} \right)$$

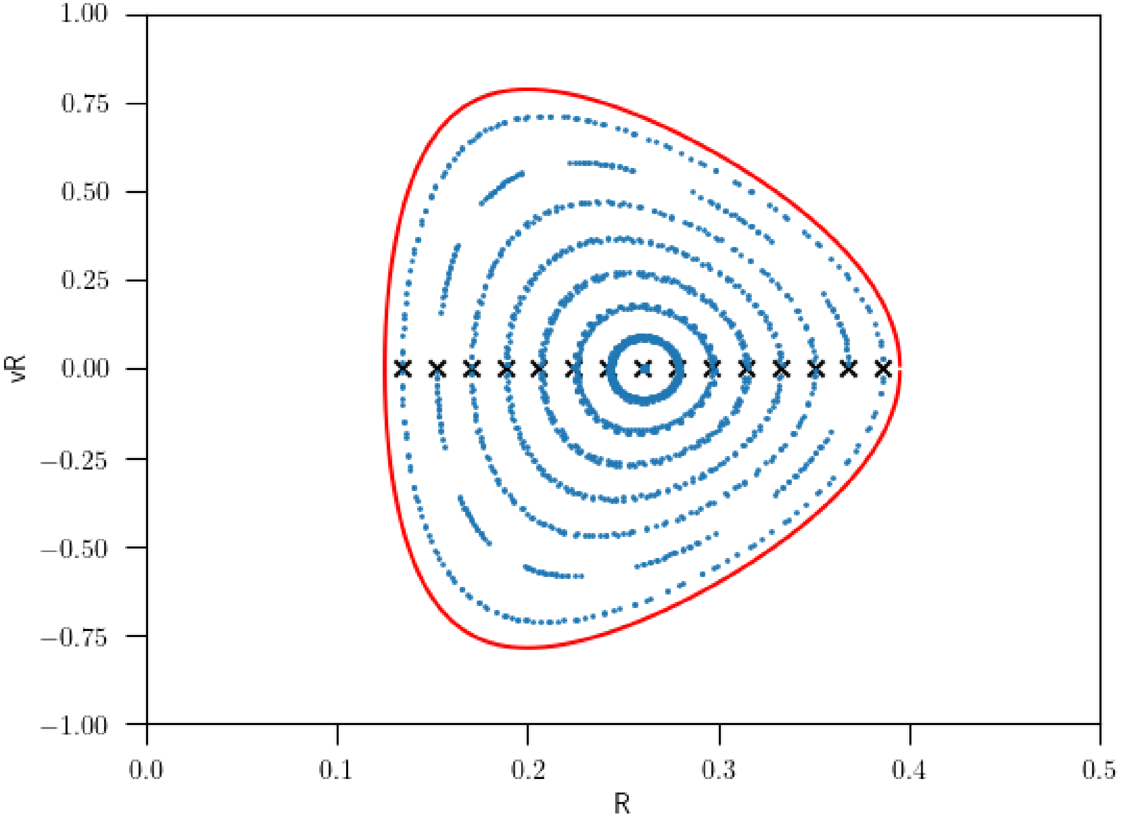


Effective Potential




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --plotpotential
```

Invariant curves : Third Integral




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 30 --nlaps 1000
```

The Third Integral I

(I is in general non analytical)

Spherical systems : $|\vec{L}| = L$ is conserved

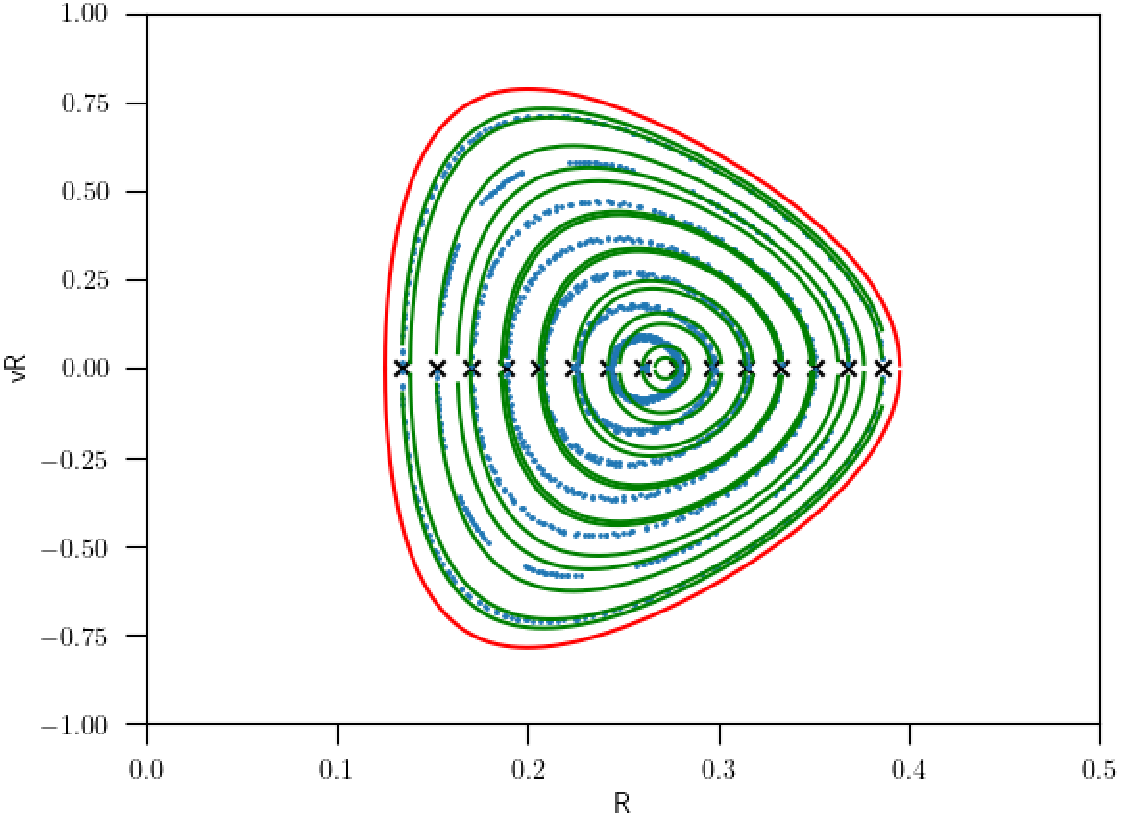
Nearly spherical potential : L is nearly an integral $= I$?

What is the curve in the Poincaré map that satisfies $L = cte$?

EXERCICE

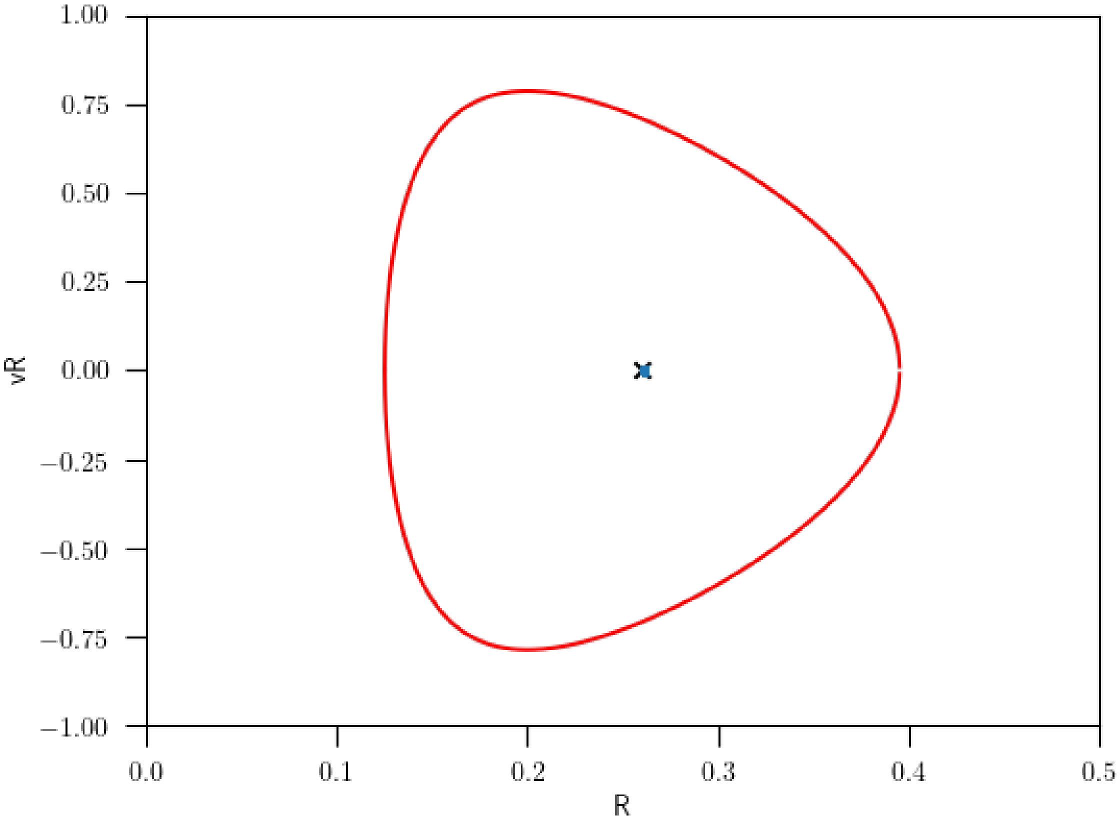
Invariant curves : Third Integral

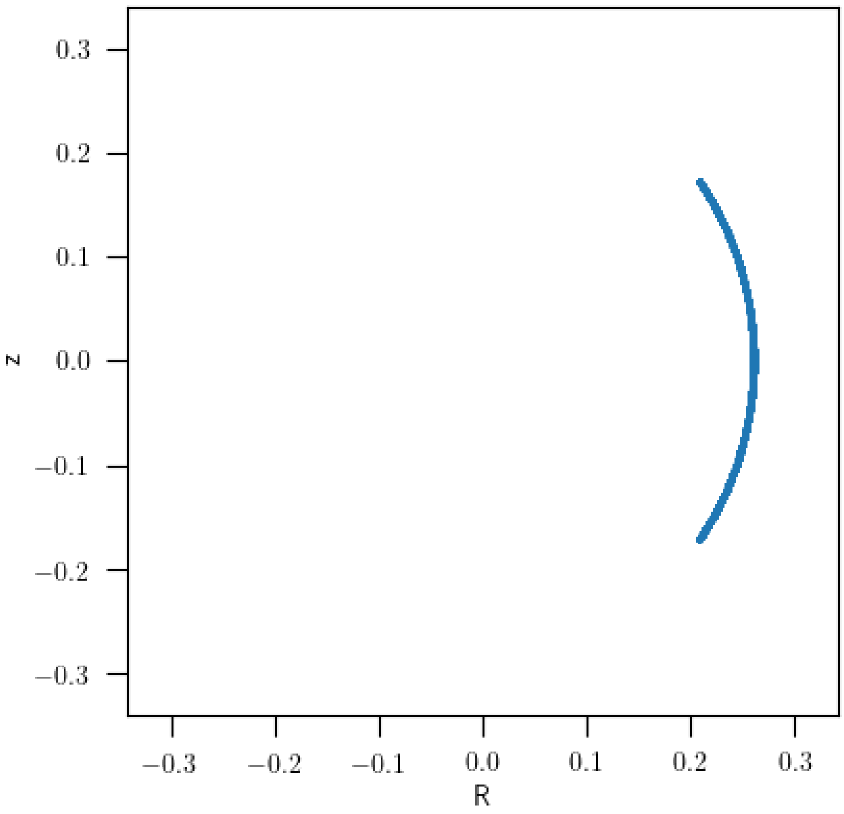
green : contours of constant total angular momentum



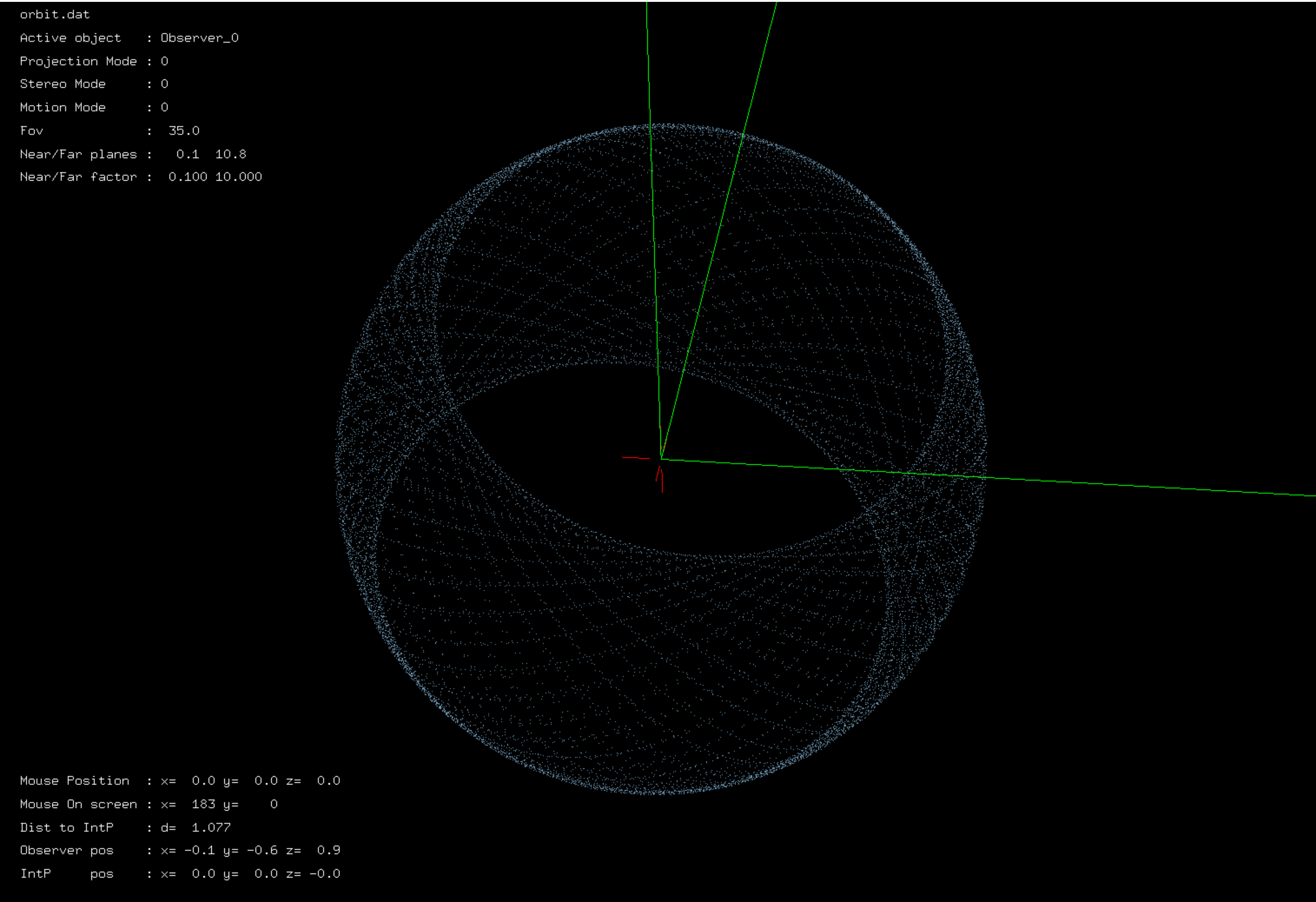

```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100 --add_IL
```

shell orbit

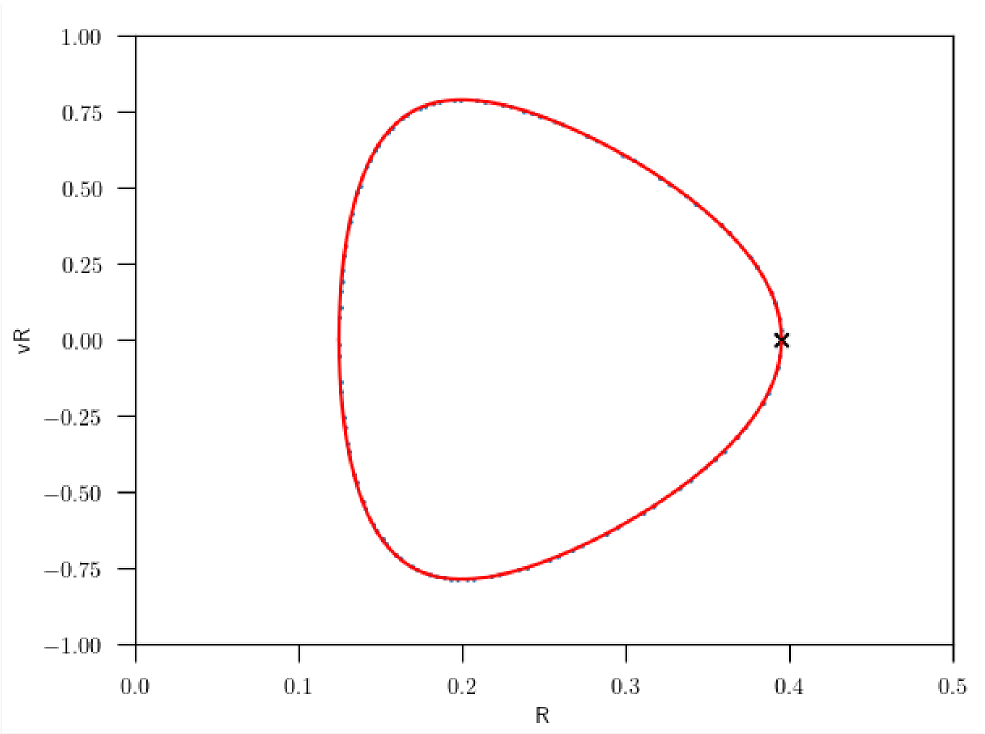


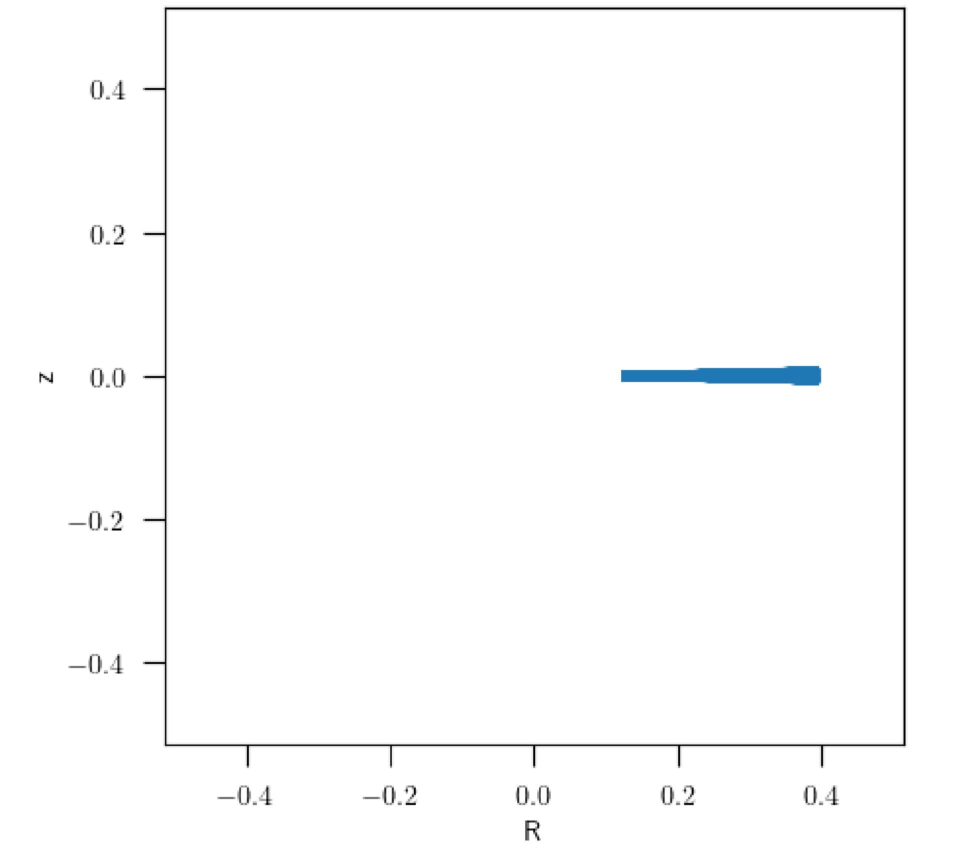



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8  --nlaps 100 --R 0.2612
```

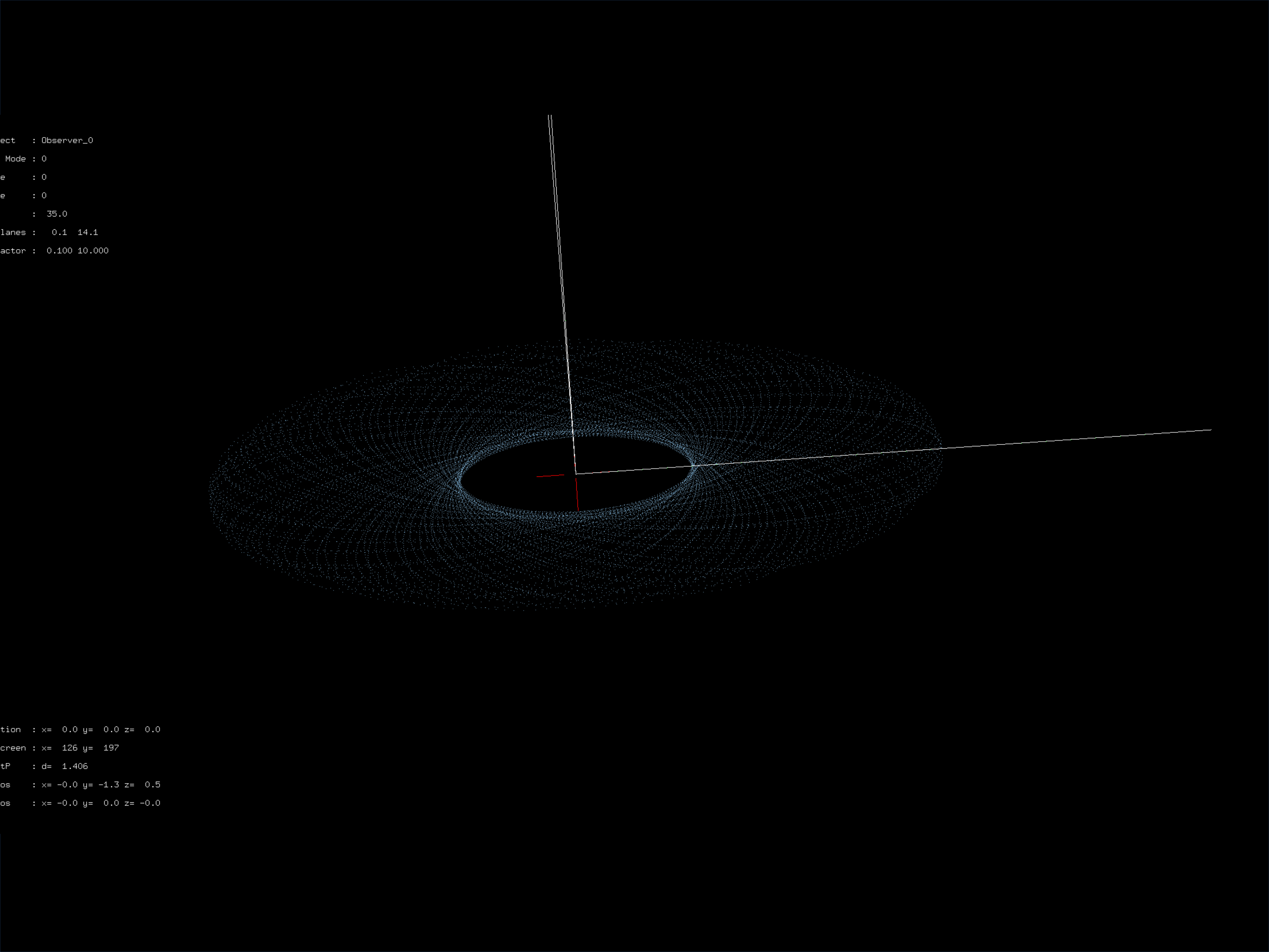

orbit.dat
Active object : Observer_0
Projection Mode : 0
Stereo Mode : 0
Motion Mode : 0
Fov : 35.0
Near/Far planes : 0.1 10.8
Near/Far factor : 0.100 10.000
Mouse Position : x= 0.0 y= 0.0 z= 0.0
Mouse On screen : x= 183 y= 0
Dist to IntP : d= 1.077
Observer pos : x= -0.1 y= -0.6 z= 0.9
IntP pos : x= 0.0 y= 0.0 z= -0.0

Large radius

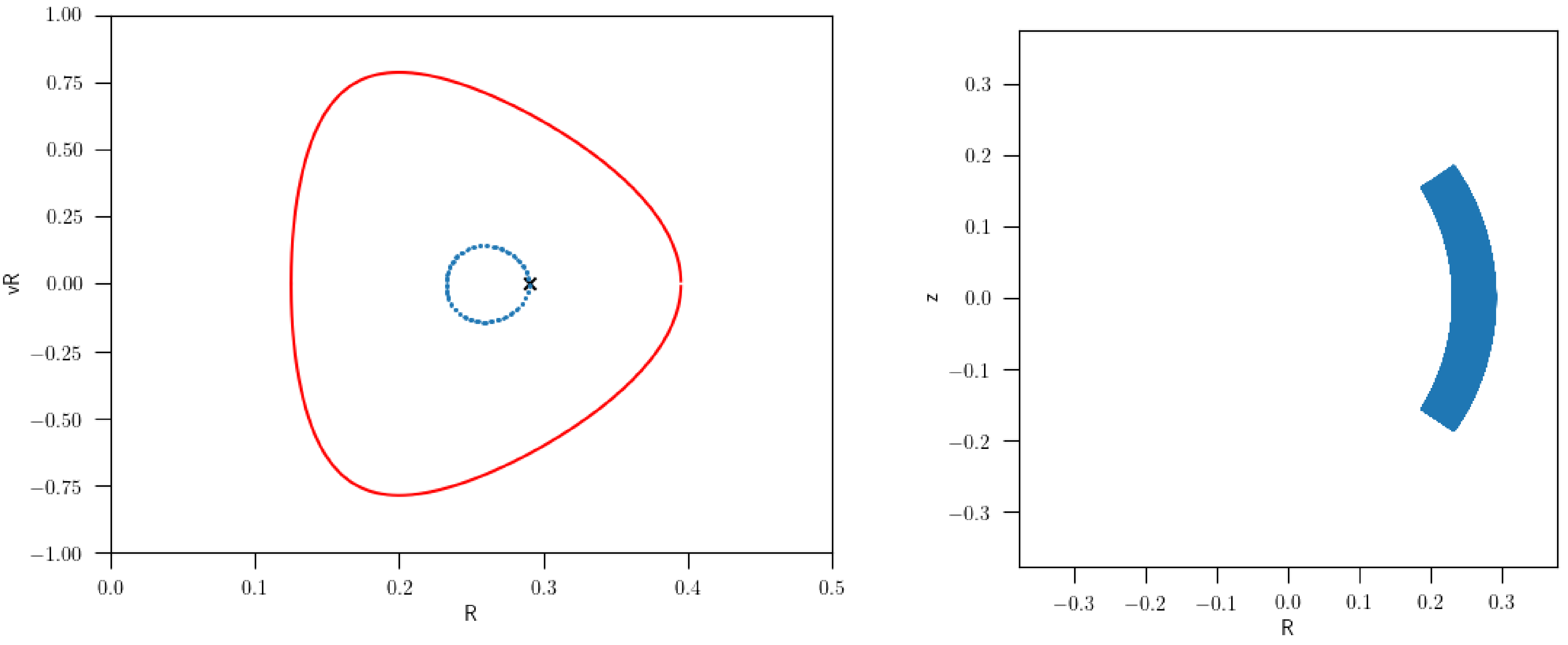




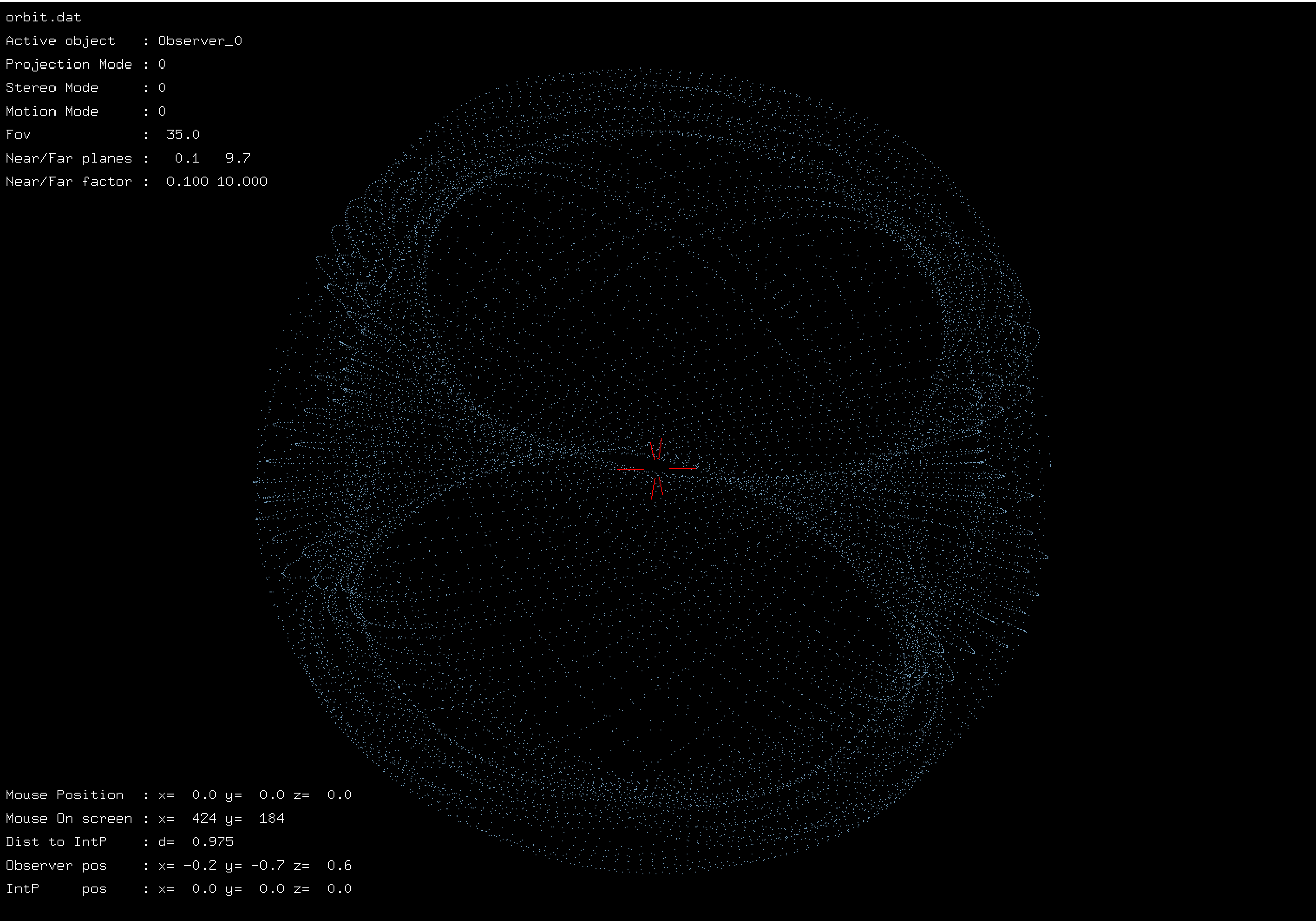

```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100 --R 0.3953
```


ect : Observer_0
Mode : 0
e : 0
e : 0
: 35.0
lanes : 0.1 14.1
actor : 0.100 10.000
tion : x= 0.0 y= 0.0 z= 0.0
creen : x= 126 y= 197
tP : d= 1.406
os : x= -0.0 y= -1.3 z= 0.5
os : x= -0.0 y= 0.0 z= -0.0

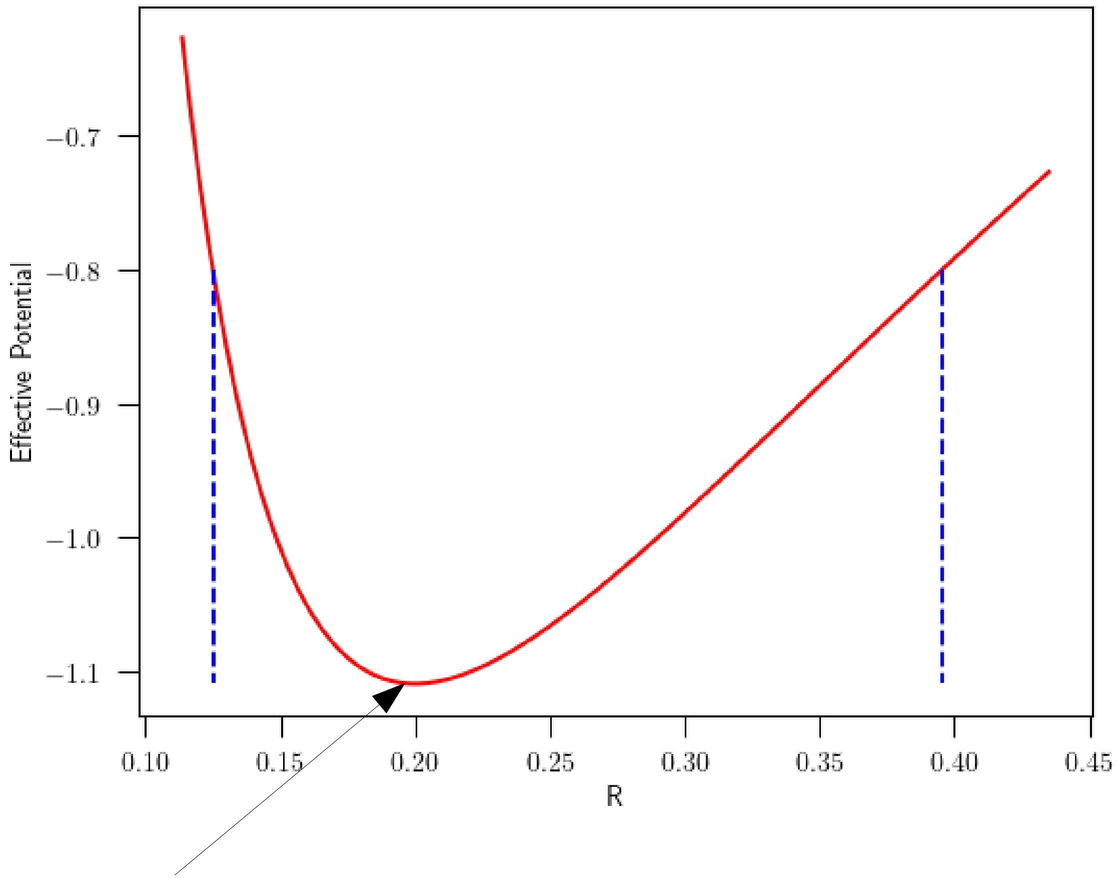
Smaller radius




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8   --norbits 15 --nlaps 100   --R 0.29
```

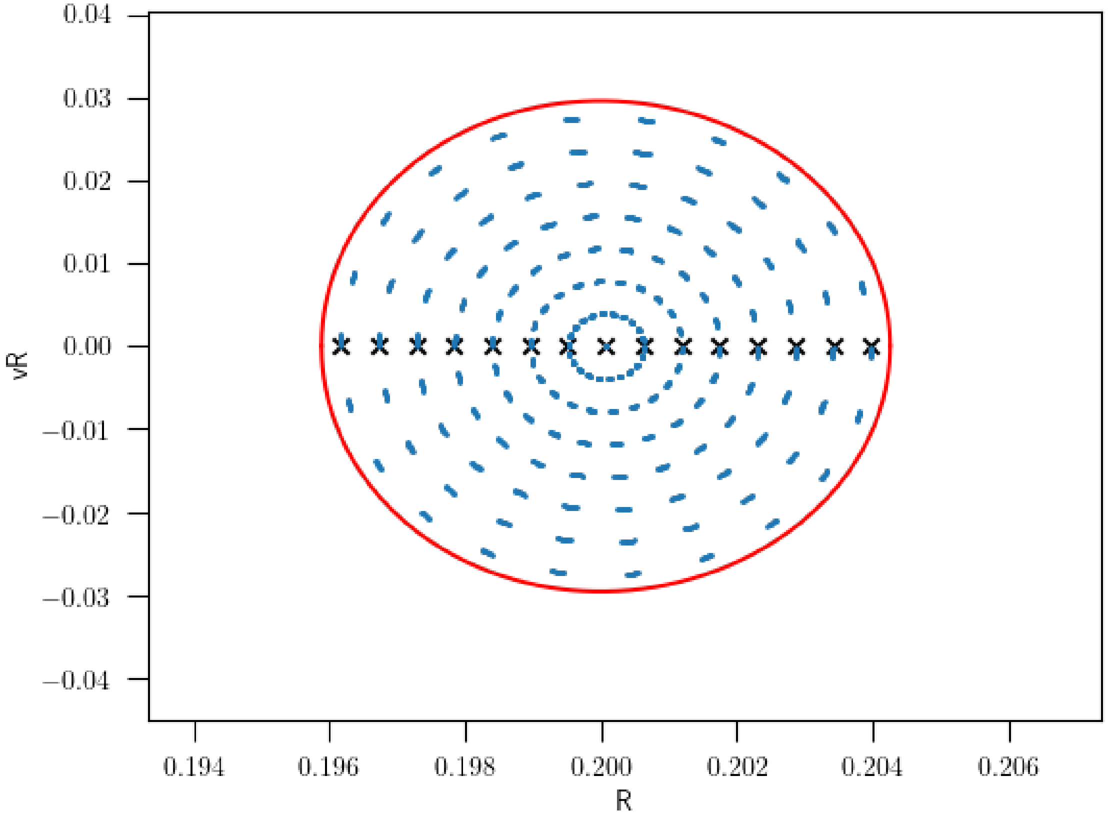

orbit.dat
Active object : Observer_0
Projection Mode : 0
Stereo Mode : 0
Motion Mode : 0
Fov : 35.0
Near/Far planes : 0.1 9.7
Near/Far factor : 0.100 10.000
Mouse Position : x= 0.0 y= 0.0 z= 0.0
Mouse On screen : x= 424 y= 184
Dist to IntP : d= 0.975
Observer pos : x= -0.2 y= -0.7 z= 0.6
IntP pos : x= 0.0 y= 0.0 z= 0.0

Exploring orbits at lower energy



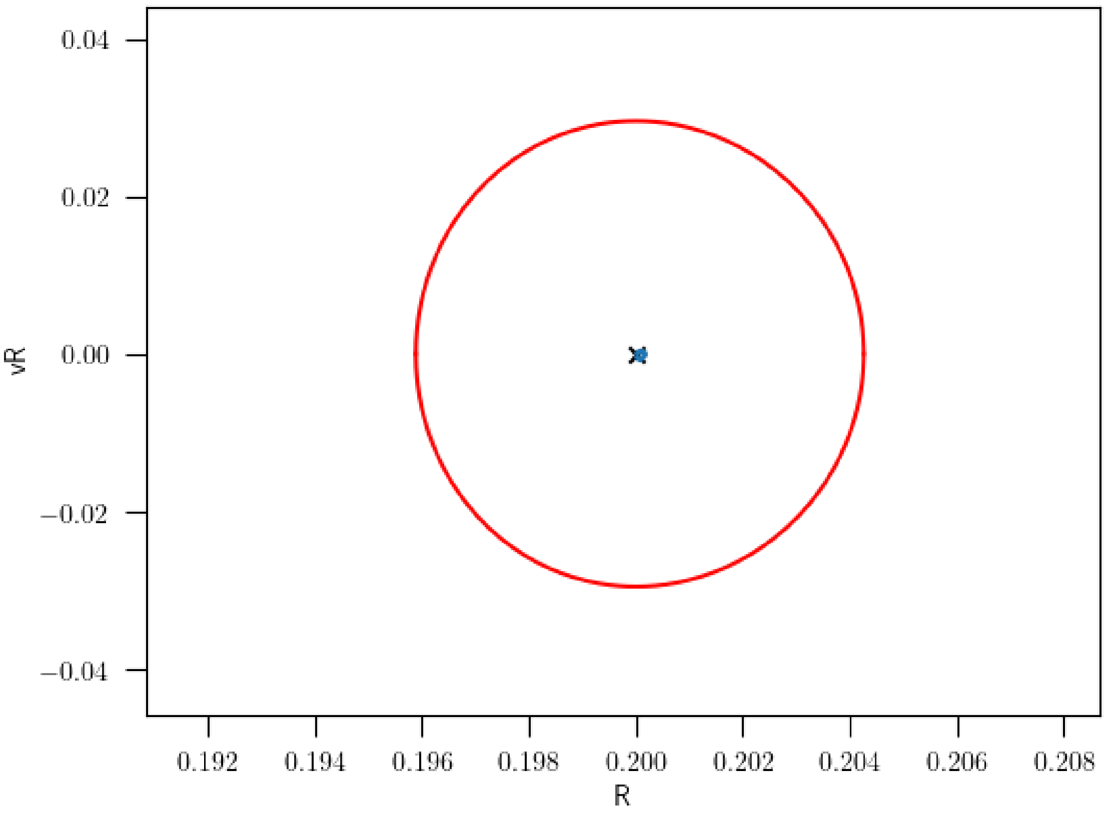

```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.109 --plotpotential
```

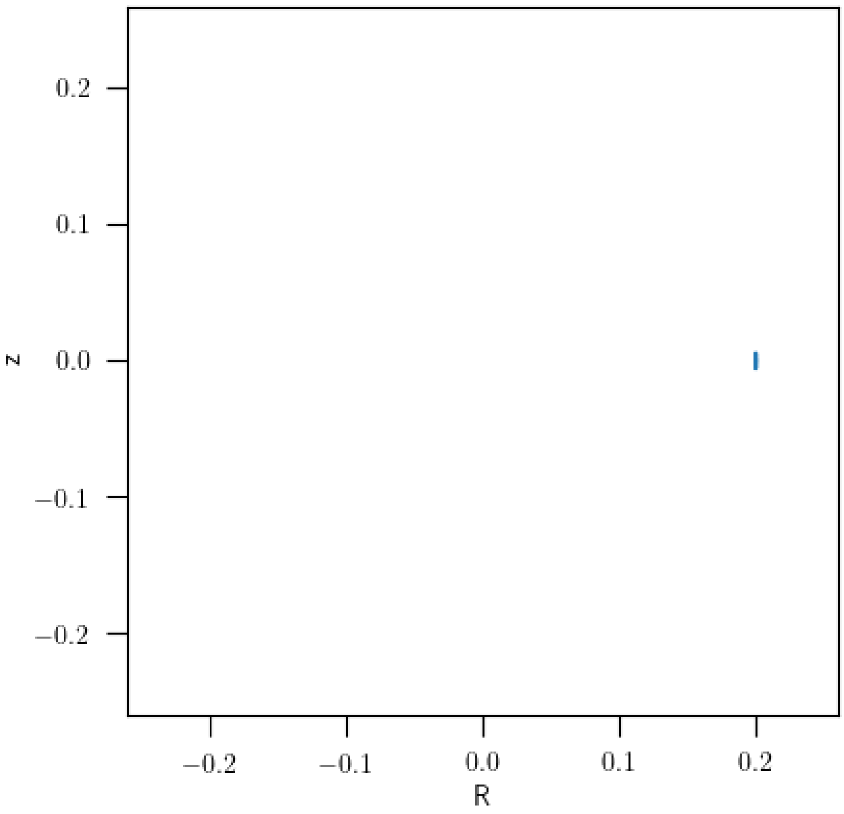
Orbits near the circular orbit energy




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.109 --norbits 15 --nlaps 100
```

Circular orbit






```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.109 --vR 0 --R 0.2   --nlaps 10
```

orbit.dat

Active object   : Observer_0

Projection Mode : 0

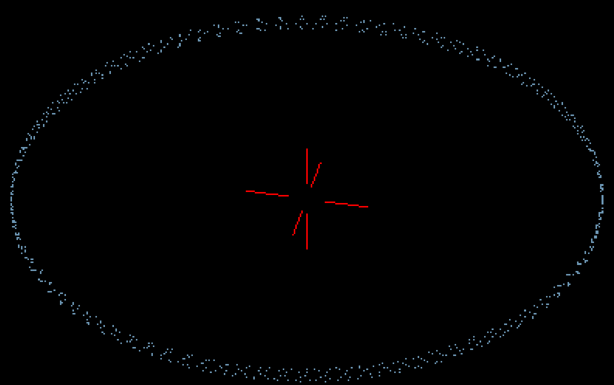
Stereo Mode   : 0

Motion Mode   : 0

Fov   : 35.0

Near/Far planes :   0.1 14.3

Near/Far factor : 0.100 10.000



Mouse Position : x= 0.0 y= 0.0 z= 0.0

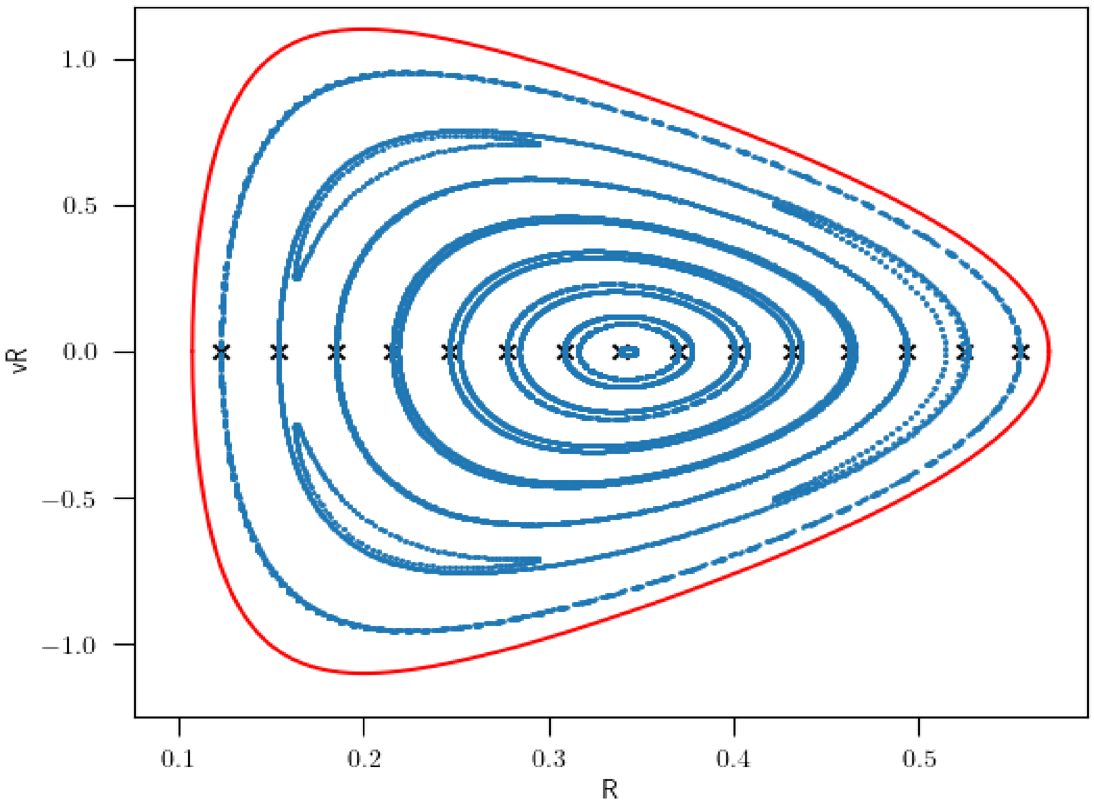
Mouse On screen : x= 247 y= -53

Dist to IntP   : d= 1.431

Observer pos   : x= 0.3 y= -1.1 z= 0.9

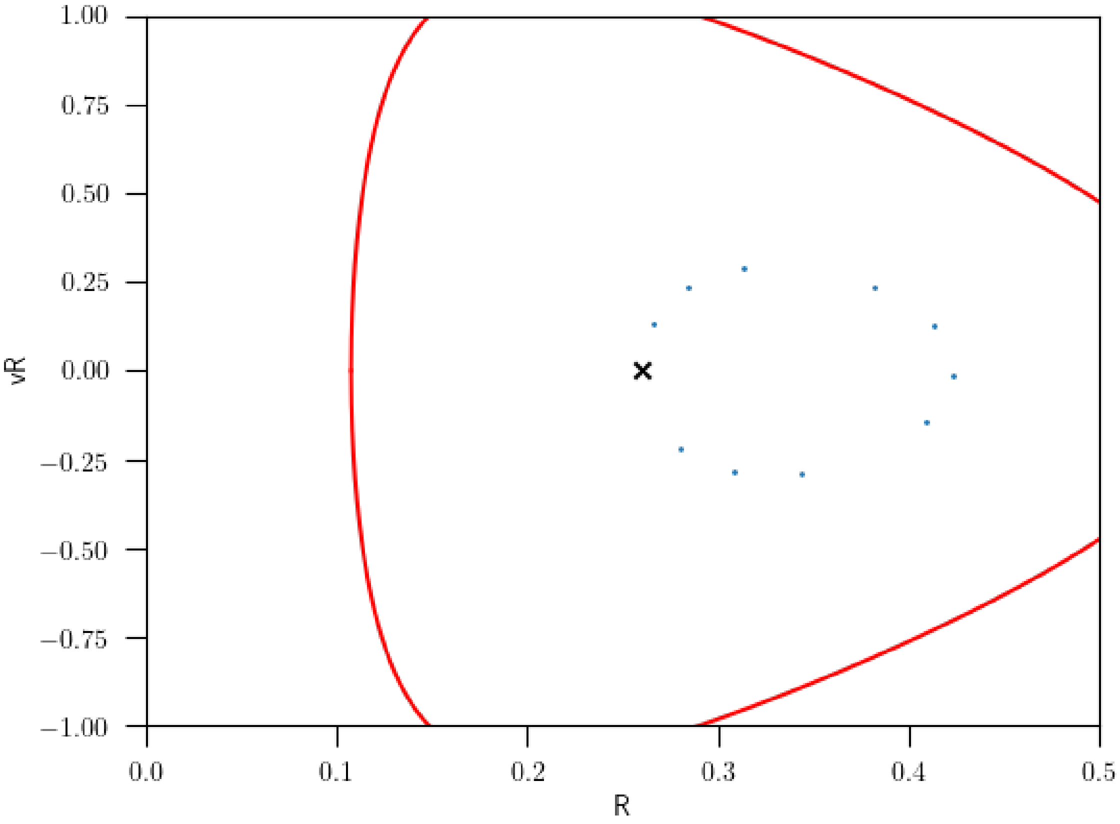
IntP   pos   : x= 0.0 y= 0.0 z= 0.0

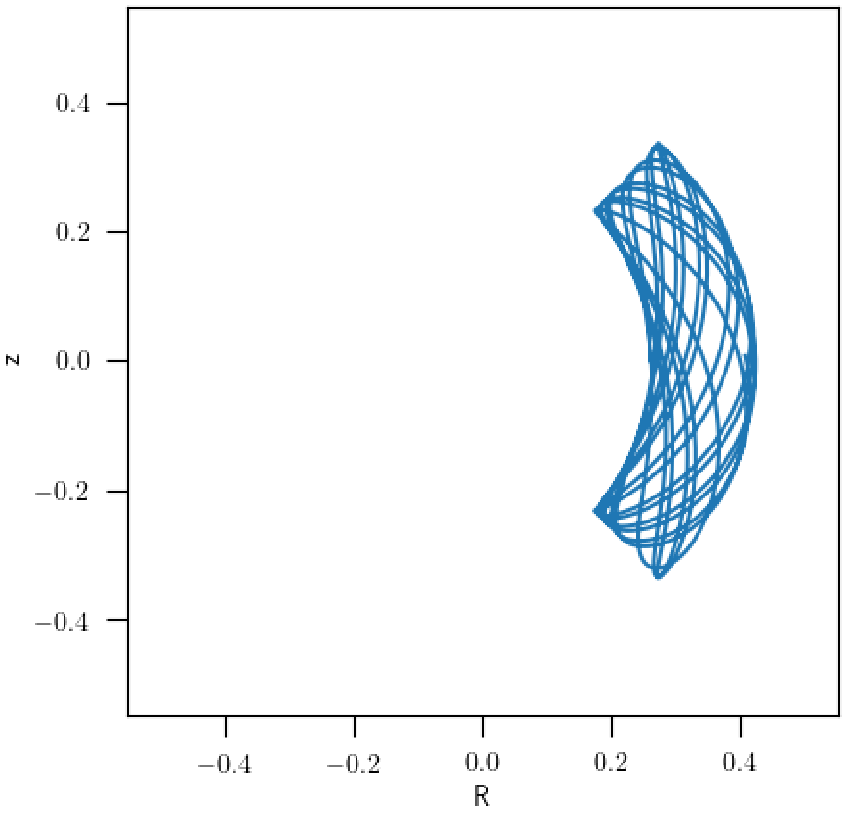
At higher energy




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --norbits 15 --nlaps 1000
```

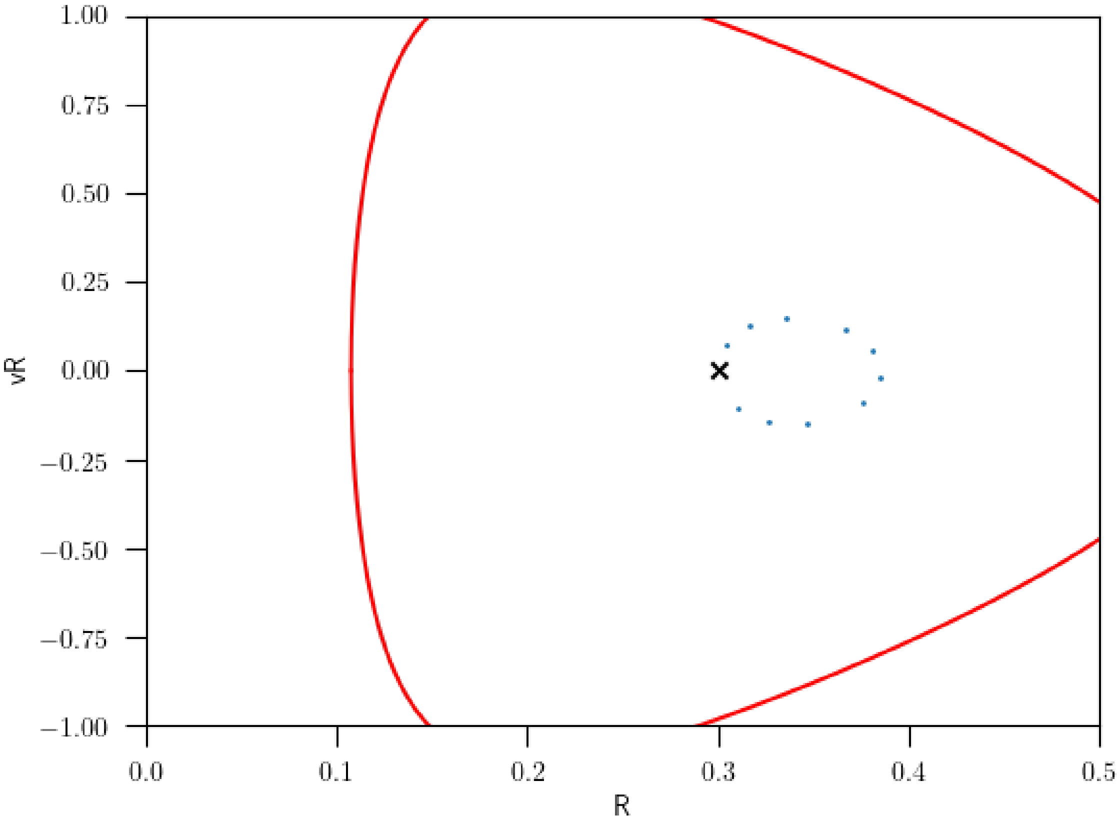
At higher energy

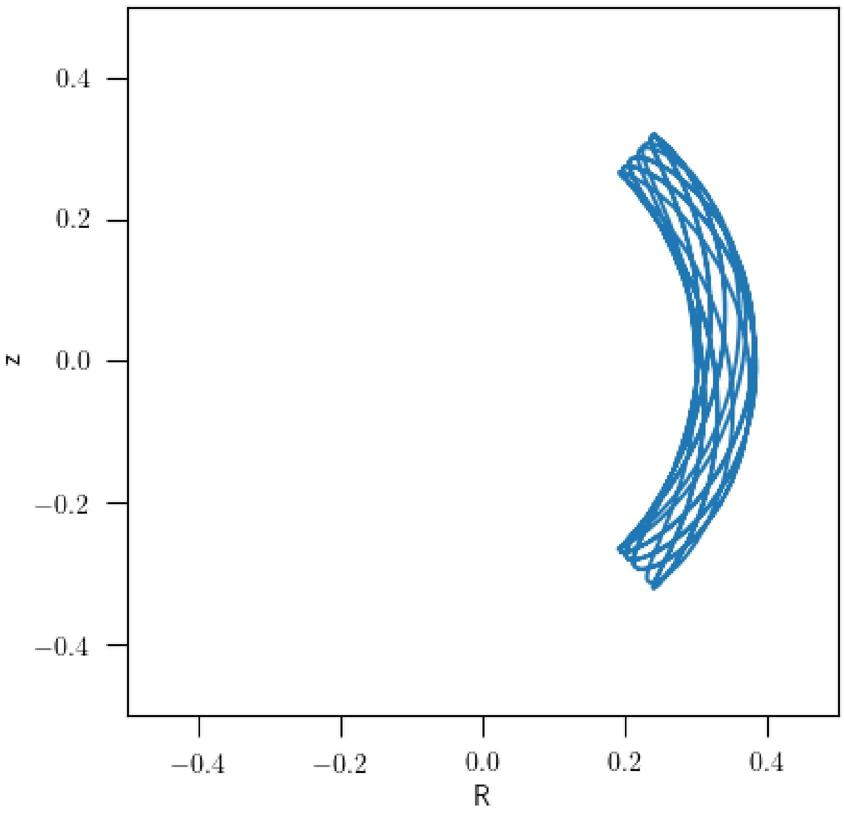





```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.26   --nlaps 10
```

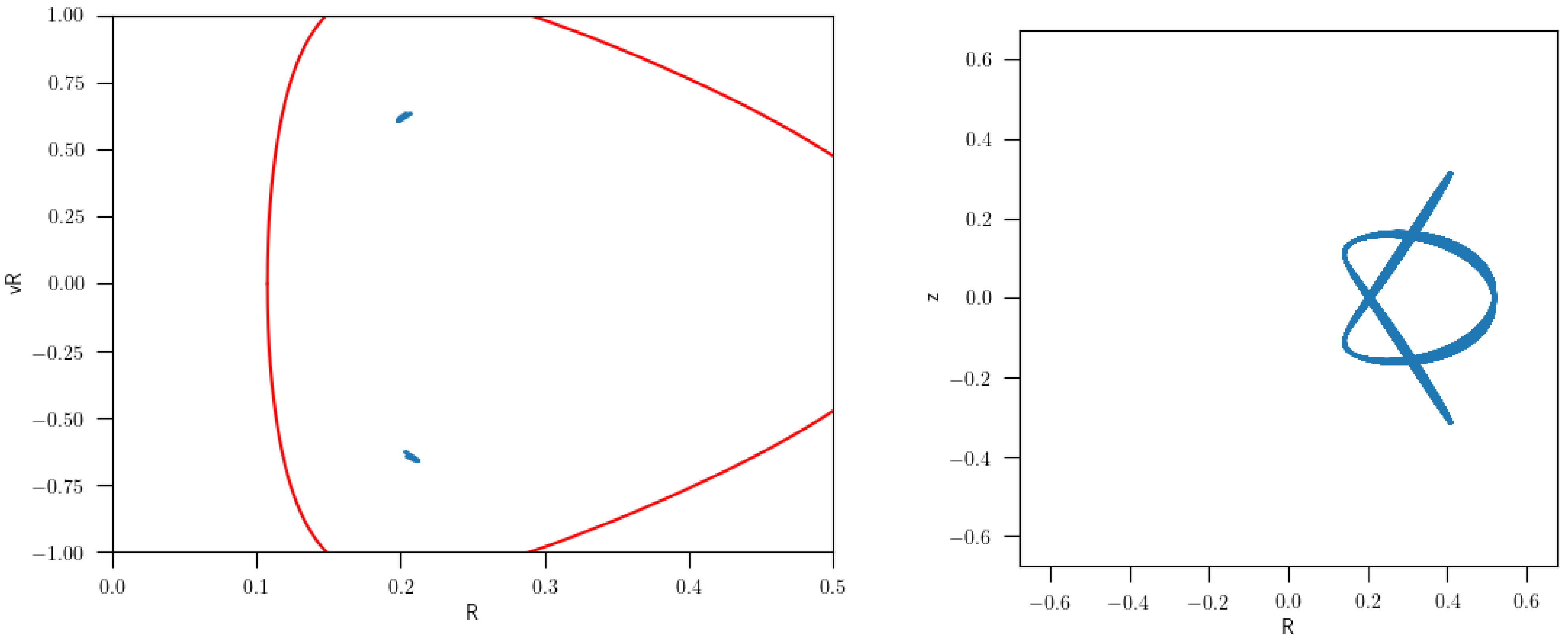
At higher energy



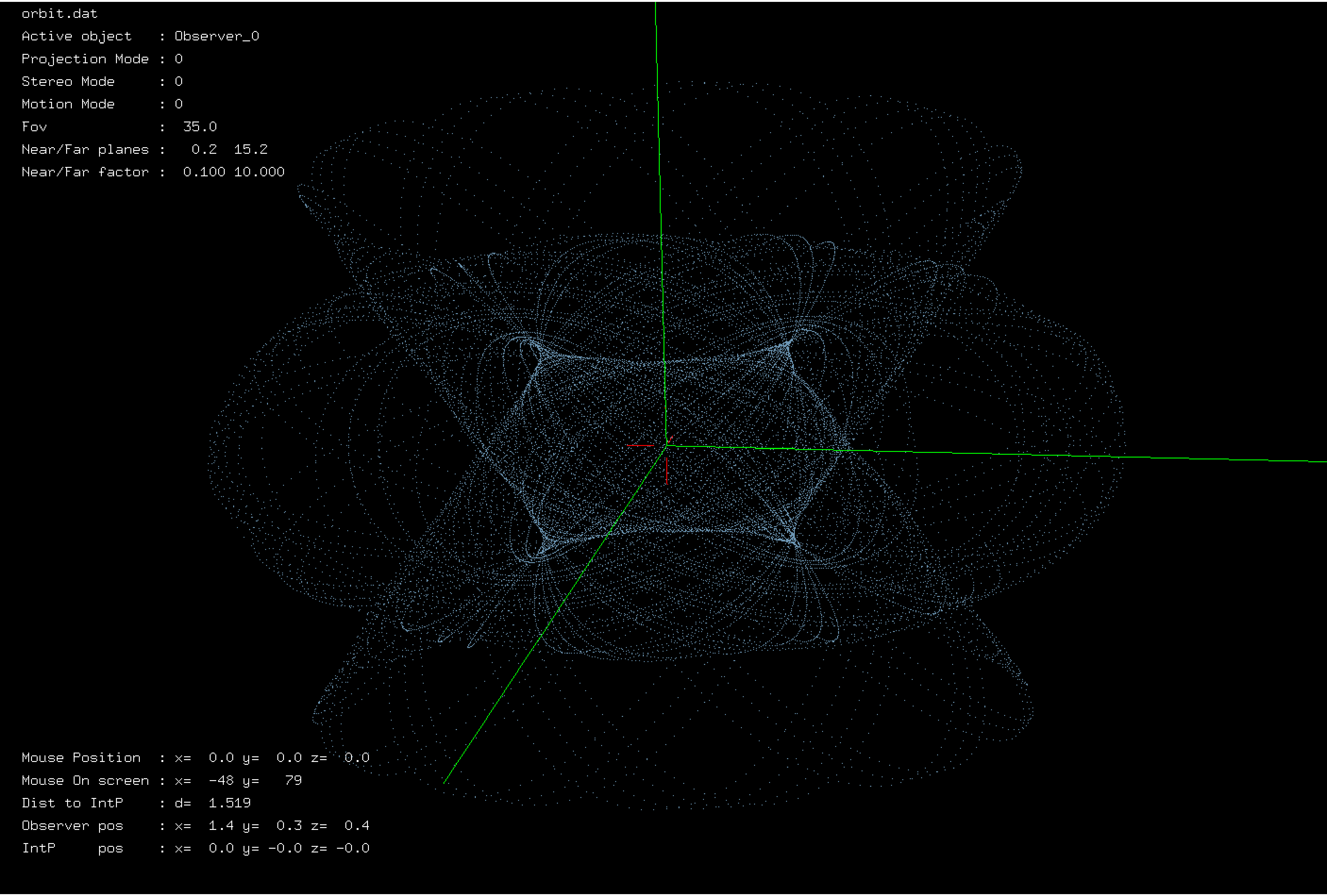



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.30   --nlaps 10
```

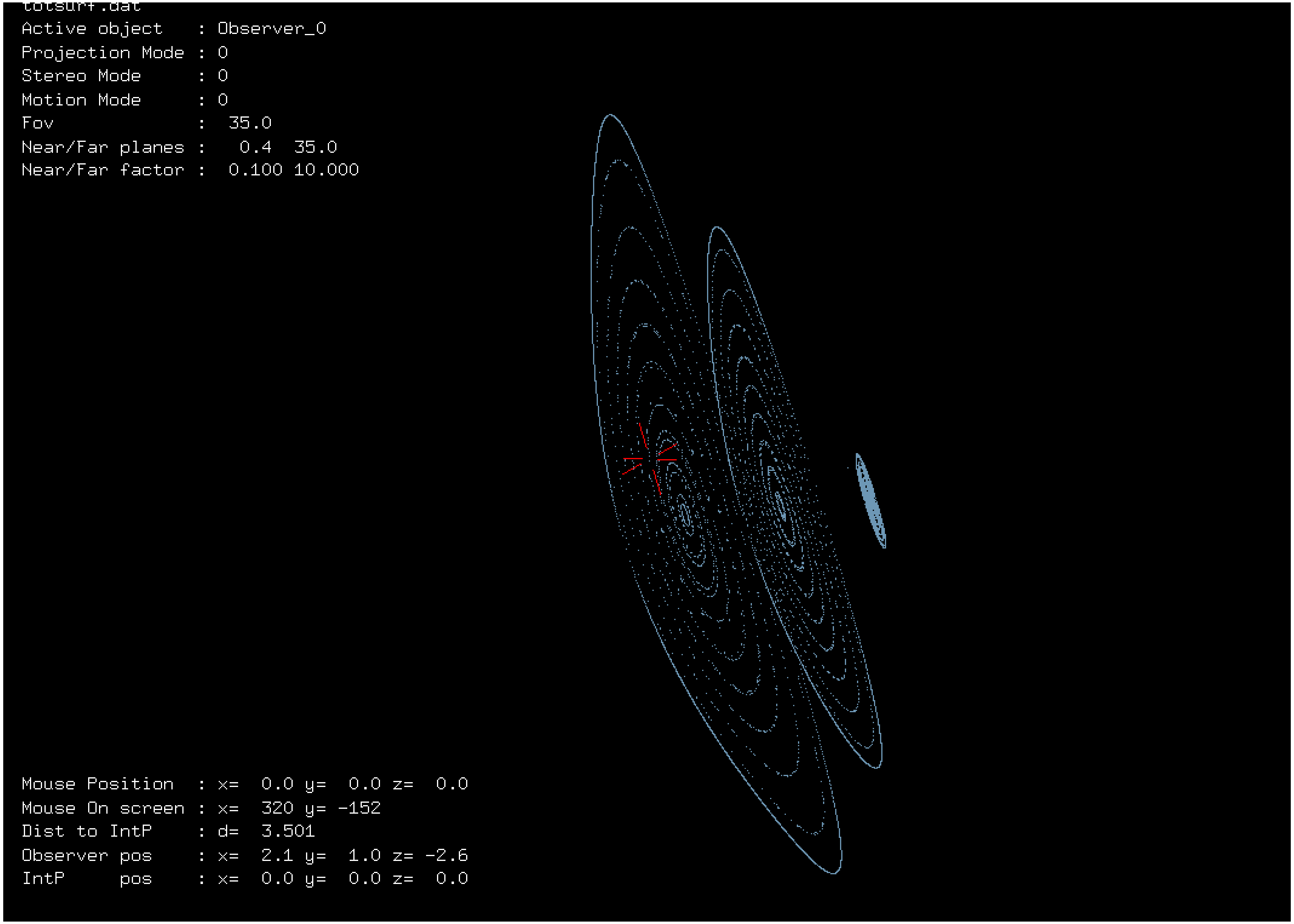
Bifurcation (resonance) : new orbit family




```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.52   --nlaps 100
```


orbit.dat
Active object : Observer_0
Projection Mode : 0
Stereo Mode : 0
Motion Mode : 0
Fov : 35.0
Near/Far planes : 0.2 15.2
Near/Far factor : 0.100 10.000
Mouse Position : x= 0.0 y= 0.0 z= 0.0
Mouse On screen : x= -48 y= 79
Dist to IntP : d= 1.519
Observer pos : x= 1.4 y= 0.3 z= 0.4
IntP pos : x= 0.0 y= -0.0 z= -0.0

Slices of different energies




```
rm surf-*.dat
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.1 --vR 0 --norbits 50
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --vR 0 --norbits 50
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --norbits 50
./concatenate.py surf-0*
glups --fullscreen -p glparameters  totsurf.dat
```

The End