Stellar orbits

4th part

Outlines

Orbits in planar non-axisymmetric potential

- Surface of sections
    - energy dependency
    - flattening dependency
- Integrals of motions

Orbits in planar non-axisymmetric rotating potential

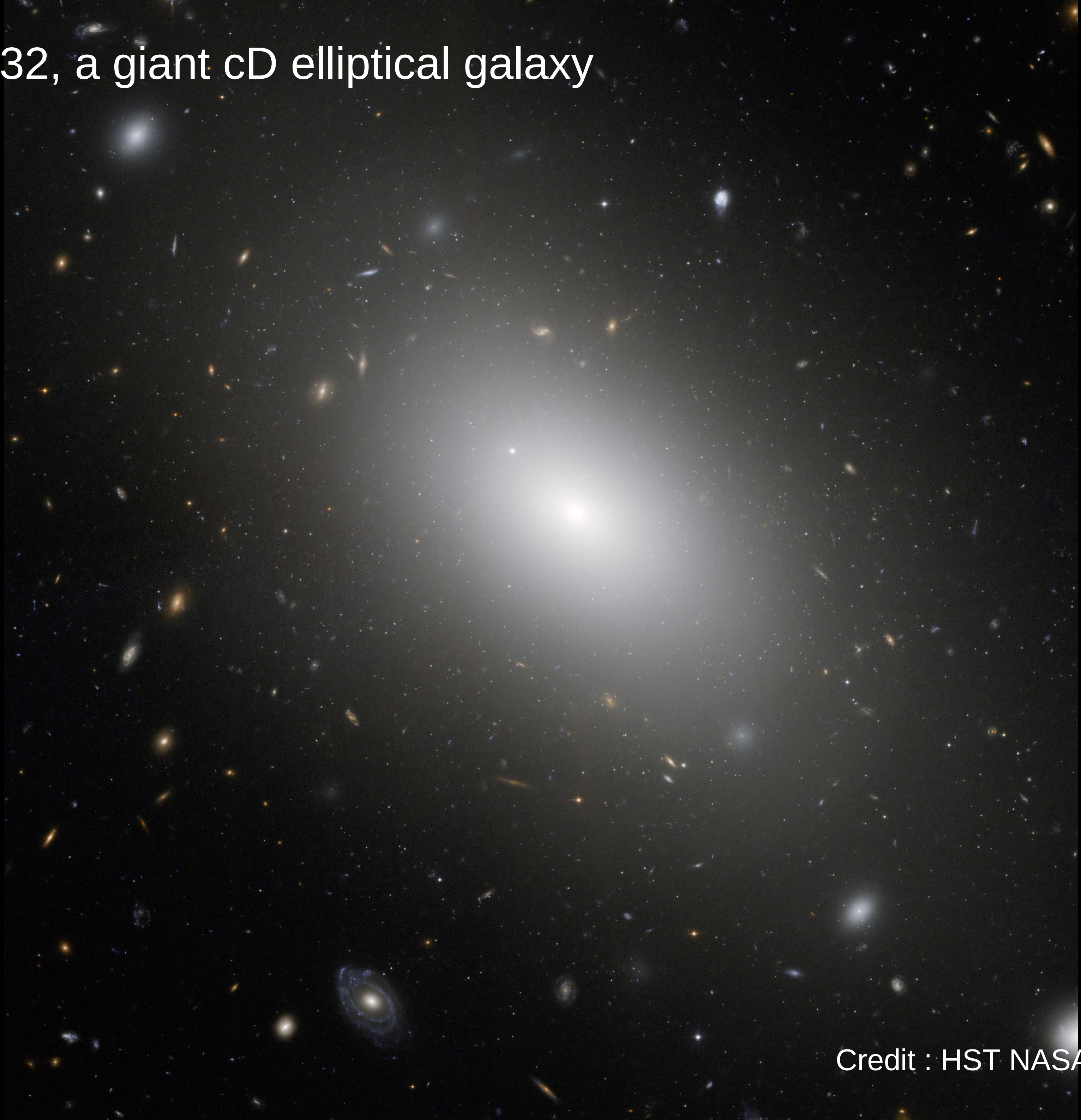
- The Jacobi integral
- Lagrange points
- Orbits around Lagrange points
- Orbits not confined to Lagrange points

Weak bars

- The Lindblad resonances
- Orbit families in realistic bars

Orbits
in planar non-axisymmetric potentials

NGC 1132, a giant cD elliptical galaxy



Credit : HST NASA/ESA

NGC 1300 SBb

Surfaces of section (in planar potentials)

Can we visualize the phase phase and check if an additional integral of motion exists ?

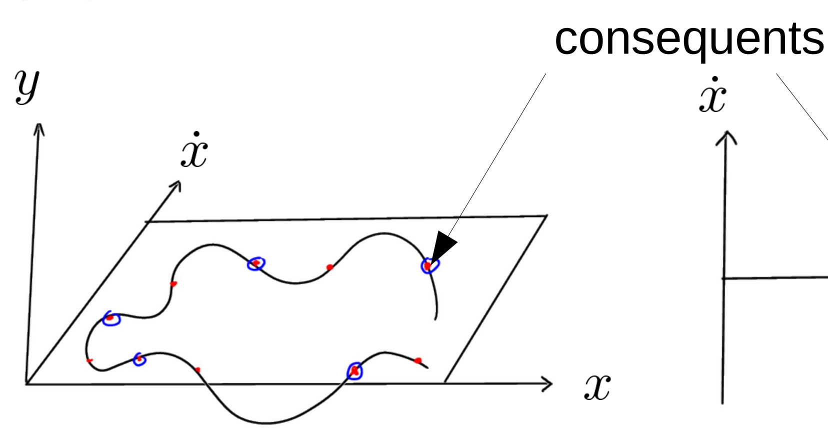
Idea :

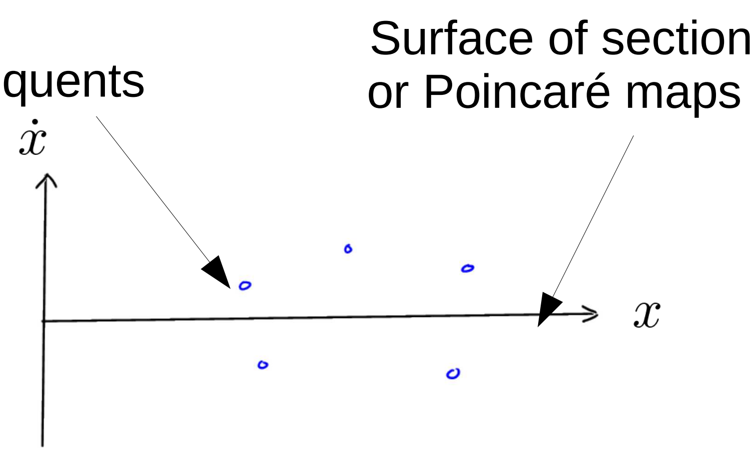
We study the orbits in the plane z=0

- 4-D — 4 indep. variables $(x, y, \dot{x}, \dot{y})$
- Energy E

→ 3-D — 3 indep. variables $(x, y, \dot{x})$

- Drawing a 3-D phase space is still not easy. Instead, we draw slices of the phase space. We plot only phase space points that:
  - cross the $y = 0$ plane
  - have $\dot{y} > 0$





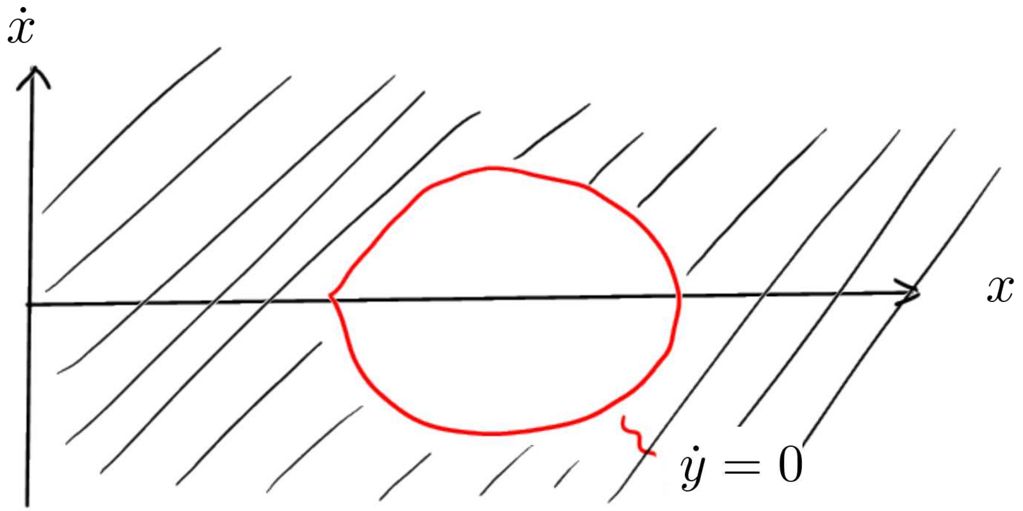
Surfaces of section (in planar potentials)

- A point in the surface of section (for a given E) defines an orbit as the three independent variables $(x, \dot{x}, y=0)$ are defined.
- Even if orbits have the same energy, they will never intersect in the plane.
- Zero velocity curve : curve defined by $\dot{y} = 0$

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \Phi(x, y=0) \qquad \Rightarrow \qquad \dot{x} \leqslant \pm\sqrt{2\left[E - \Phi(x, y=0)\right]}$$

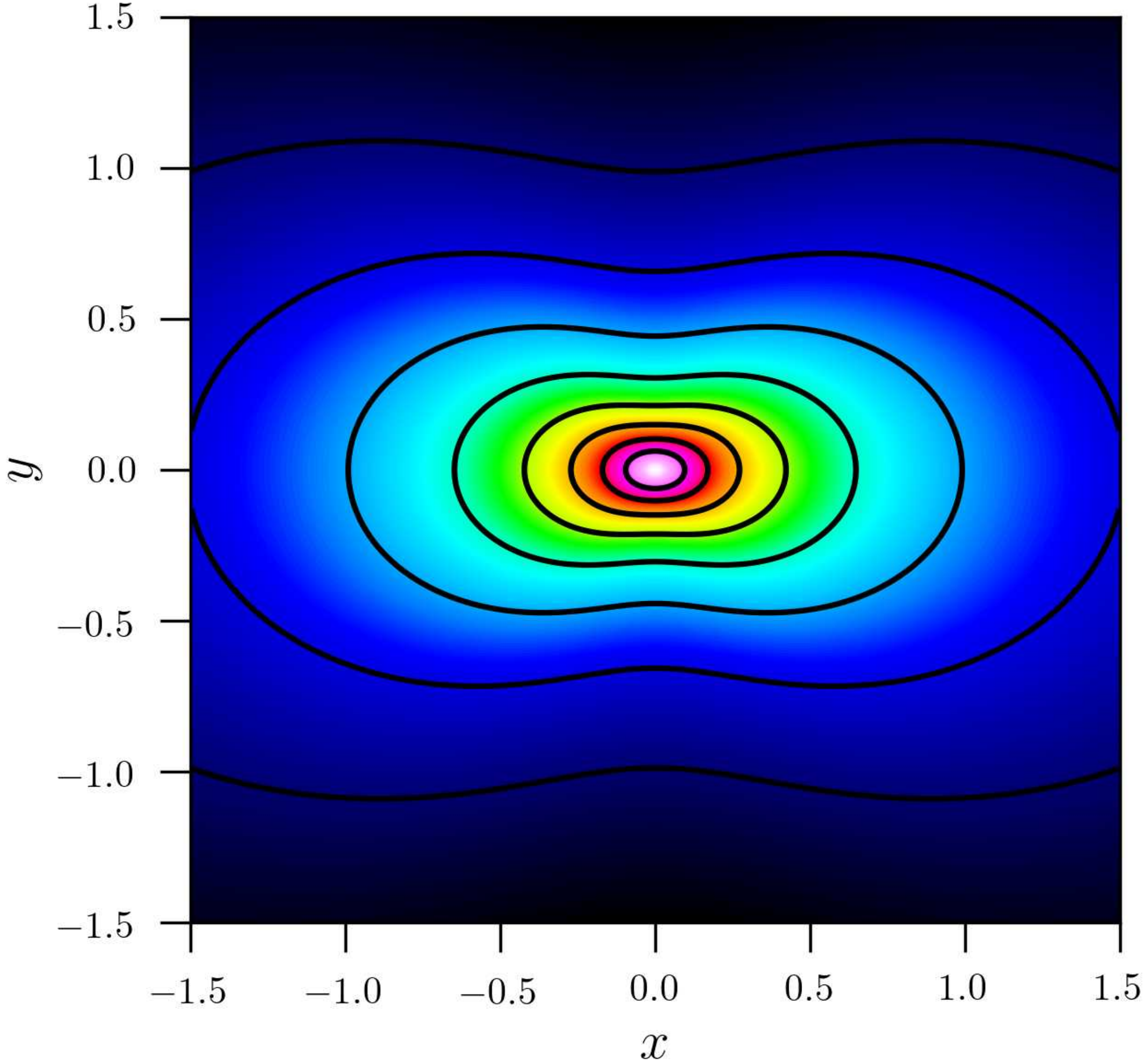
$$\dot{x}(x) = \pm\sqrt{2\left[E - \Phi(x, y=0)\right]}$$

defines the accessible region of the phase space



Bar model : Logarithmic potential:
Vo=1 Rc=0.13 q=0.8)

$$\Phi_{\log}(x,y) = \frac{1}{2}V_0^2 \ln\left(R_c^2 + x^2 + \left(\frac{y}{q}\right)^2\right)$$



$$R \ll R_c$$

Orbits in planar non-axisymmetric static potential

Model : logarithmic potential

$$\phi(x,y) = \frac{1}{2} V_0^2 \ln\left(R_c + x^2 + \frac{y^2}{q^2}\right)$$

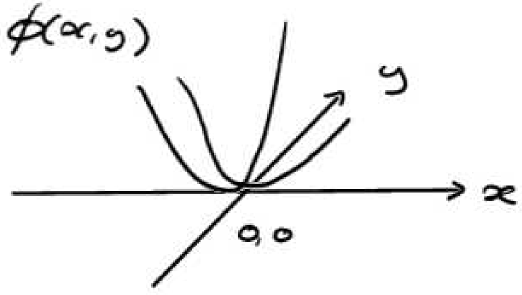
q : flattening parameter
(equipotential axis ratio)

Motions for $R \ll R_c$

$$\phi(x,y) \simeq \phi(0,0) + \underbrace{\frac{\partial\phi}{\partial x}\bigg|_{0,0}}_{=0} x + \underbrace{\frac{\partial\phi}{\partial y}\bigg|_{0,0}}_{=0} y + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}\bigg|_{0,0} x^2 + \frac{1}{2}\frac{\partial^2\phi}{\partial y^2}\bigg|_{0,0} y^2$$

$$\frac{\partial^2\phi}{\partial x^2}\bigg|_{0,0} = \frac{V_0^2}{R_c^2}$$

$$\frac{\partial^2\phi}{\partial y^2}\bigg|_{0,0} = \frac{V_0^2}{R_c^2}\frac{1}{q^2}$$



Equations of motion

$$\ddot{x} = -\frac{\partial \phi}{\partial x} \qquad \ddot{y} = -\frac{\partial \phi}{\partial y}$$

$\rightarrow$

$$\ddot{x} = -\frac{V_0^2}{R_c^2}\, x \qquad \ddot{y} = -\frac{V_0^2}{q^2 R_c^2}\, y$$

$$\omega_x = \frac{V_0}{R_c} \qquad \omega_y = \frac{V_0}{q R_c}$$

2 decoupled harmonic oscillators with different frequencies

$$\omega_y = \frac{1}{q}\,\omega_x \quad (q < 1)$$

if $q = \frac{n}{m}$ $\quad n, m \in \mathbb{N}$

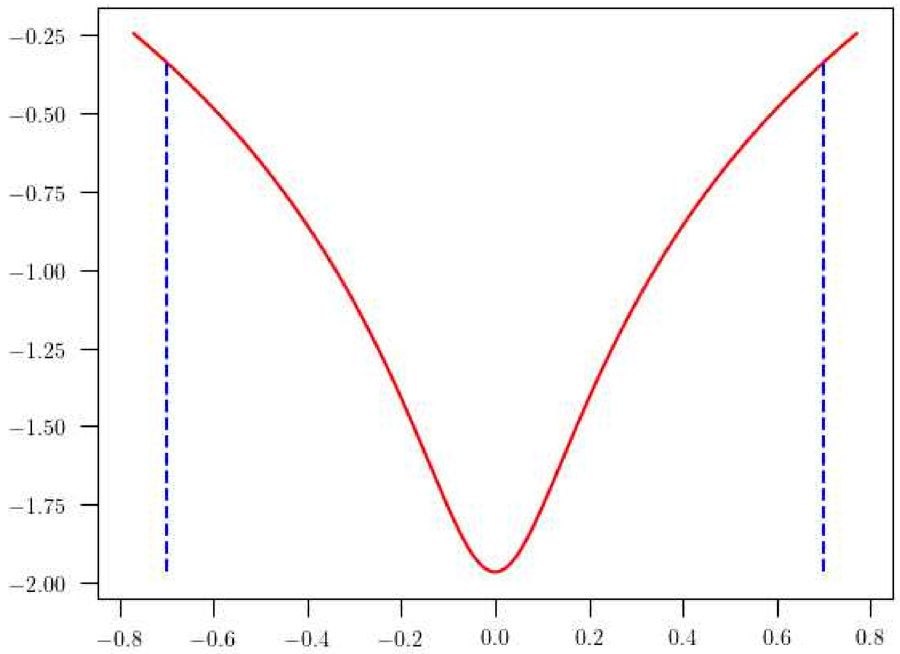
$\Rightarrow$ closed orbit

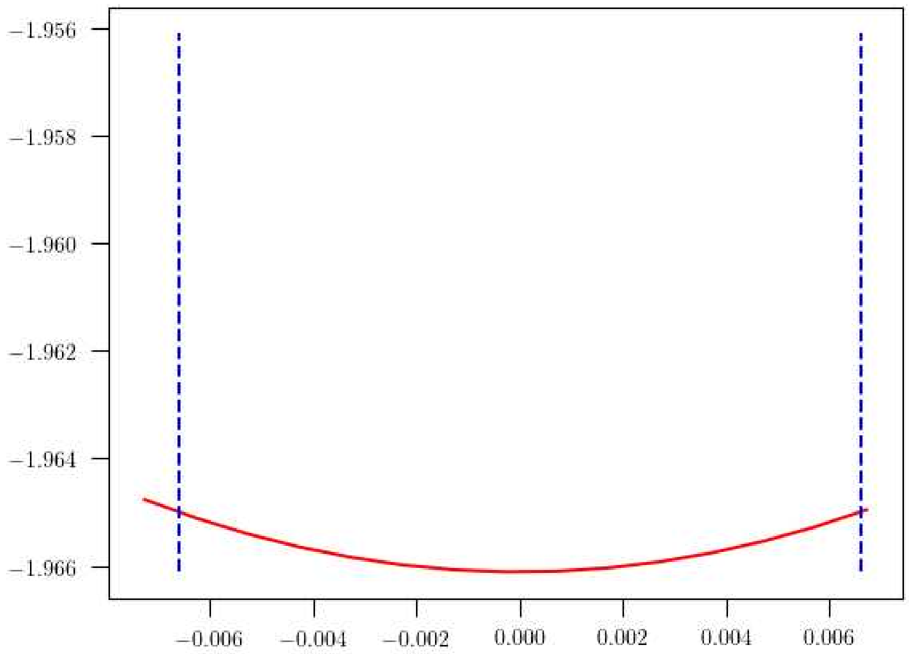
Integrals of motions (Hamiltonians)

$$H_x = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_x^2 x^2$$

$$H_y = \frac{1}{2}\dot{y}^2 + \frac{1}{2}\omega_y^2 y^2$$

Potential and energy



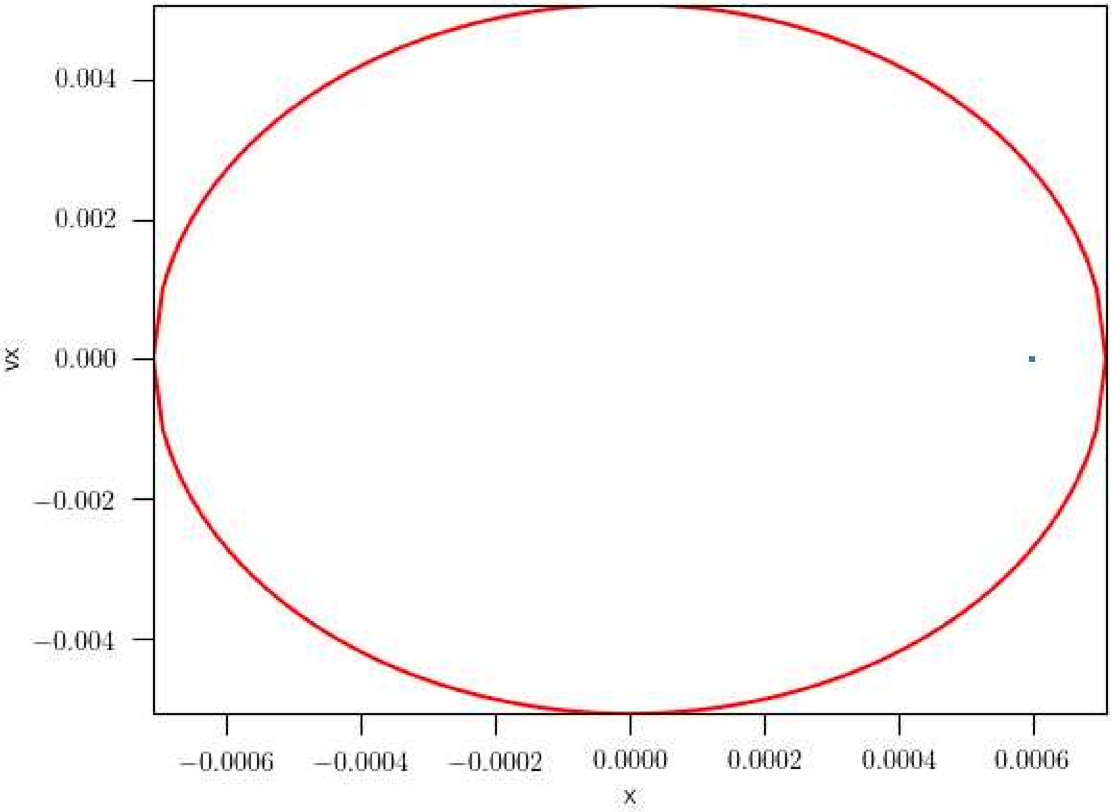


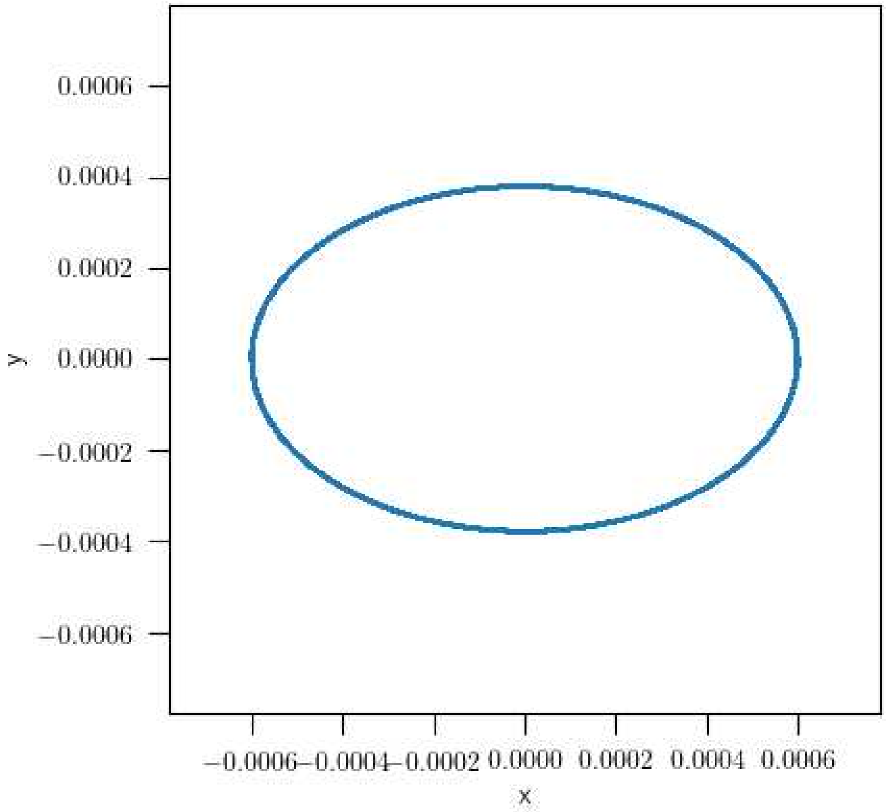
$R << R_c$

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --plotpotential
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.965  --plotpotential
```

The flattening – frequency dependency

$$q = 1$$

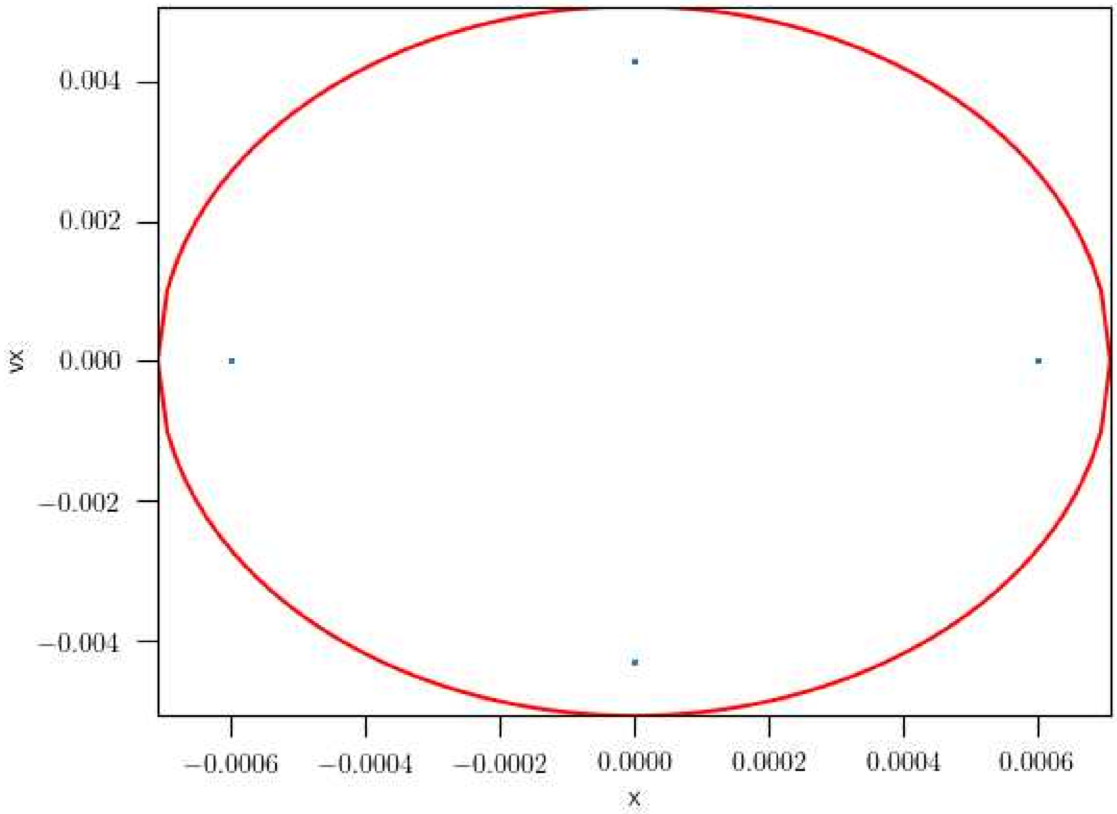


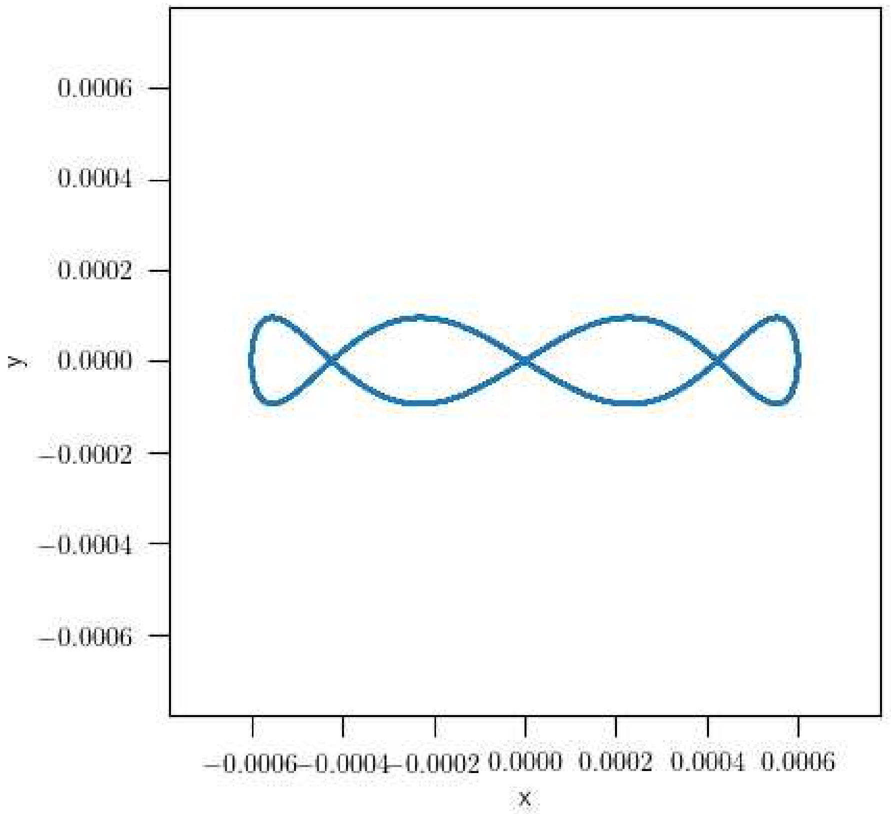



```
./mapping.py --V0 1. --Rc 0.14 --q 1.0   -E -1.9661   --x 0.0006
```

The flattening – frequency dependency

$$q = 0.25$$

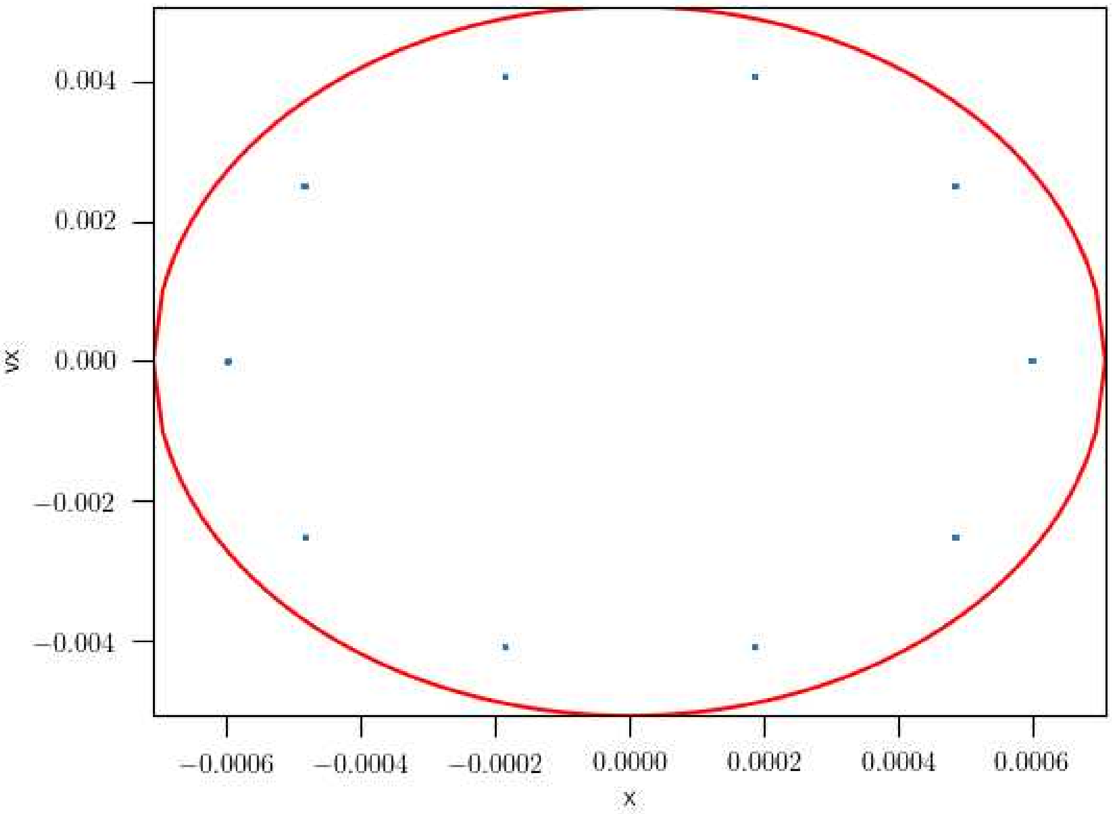


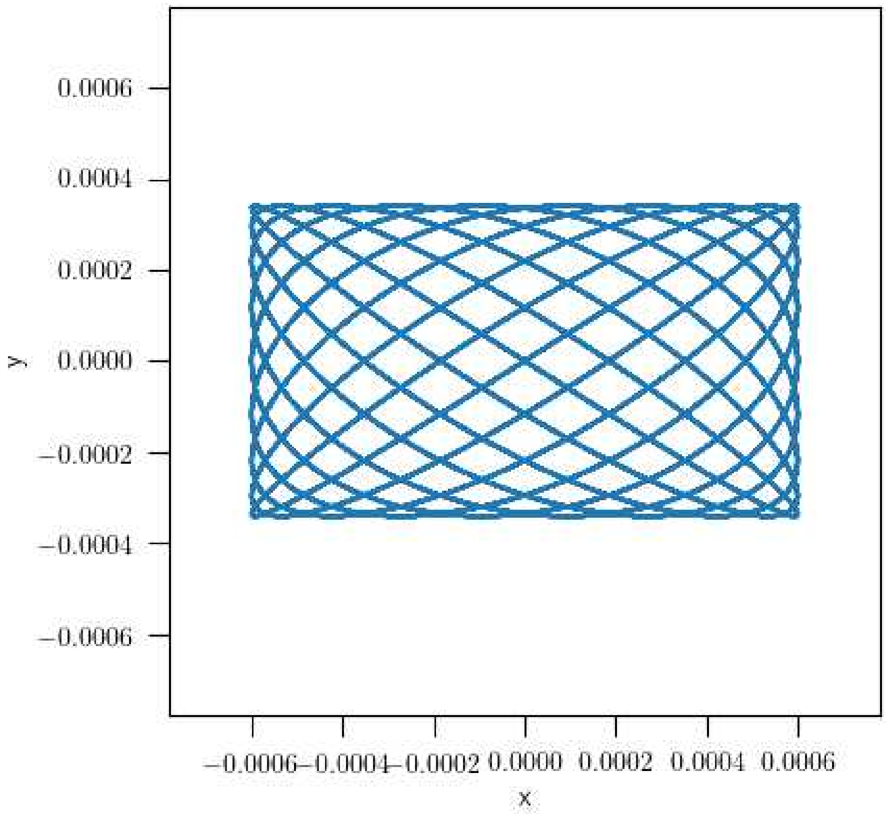



```
./mapping.py --V0 1. --Rc 0.14 --q 0.25   -E -1.9661   --x 0.0006
```

The flattening – frequency dependency

$$q = 0.9$$

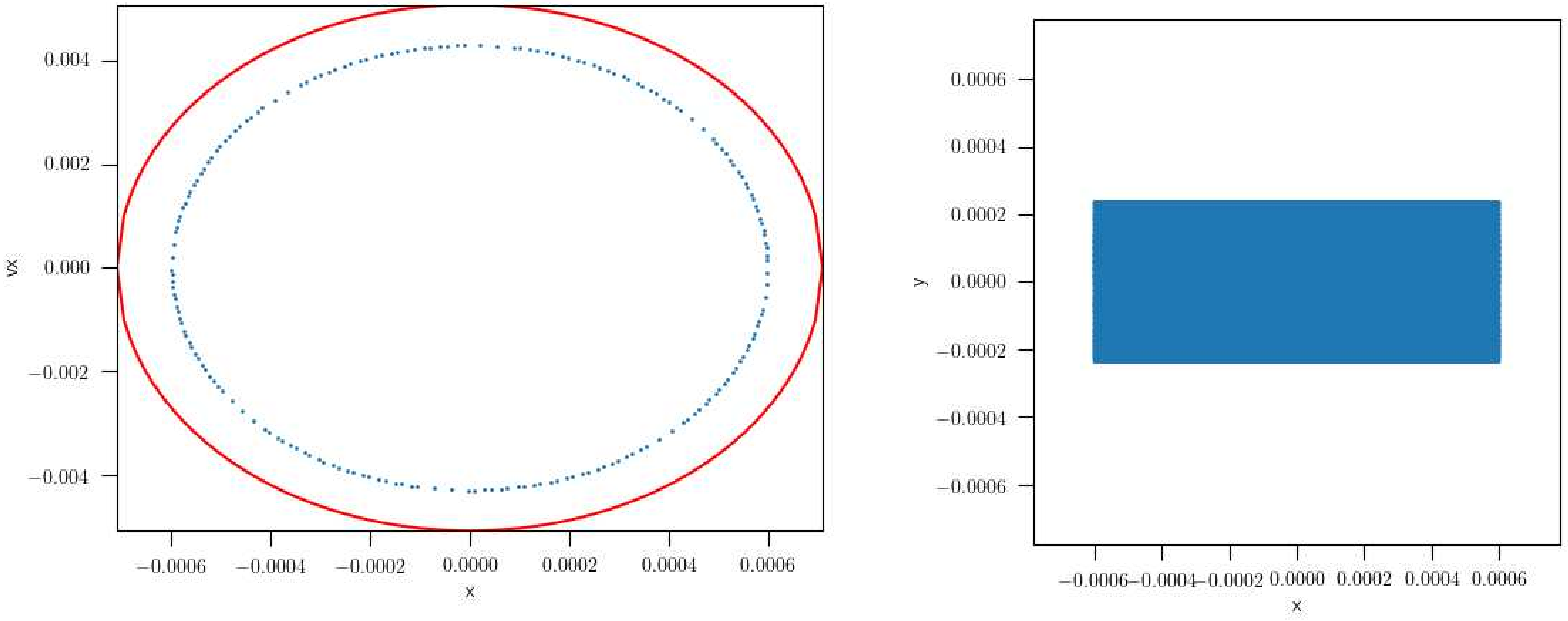





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9   -E -1.9661   --x 0.0006
```

The flattening – frequency dependency

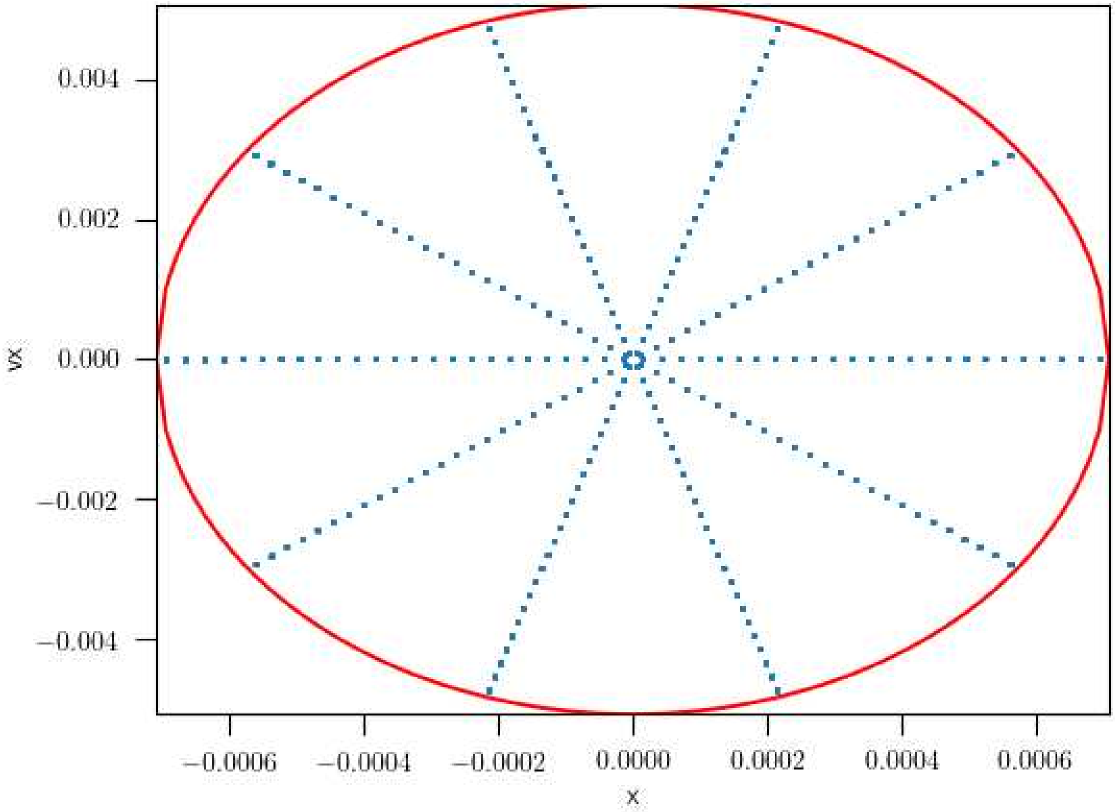
$$q = 0.62388462341$$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.62388462341 -E -1.9661 --x 0.0006 --nlaps 200
```

Complete phase space

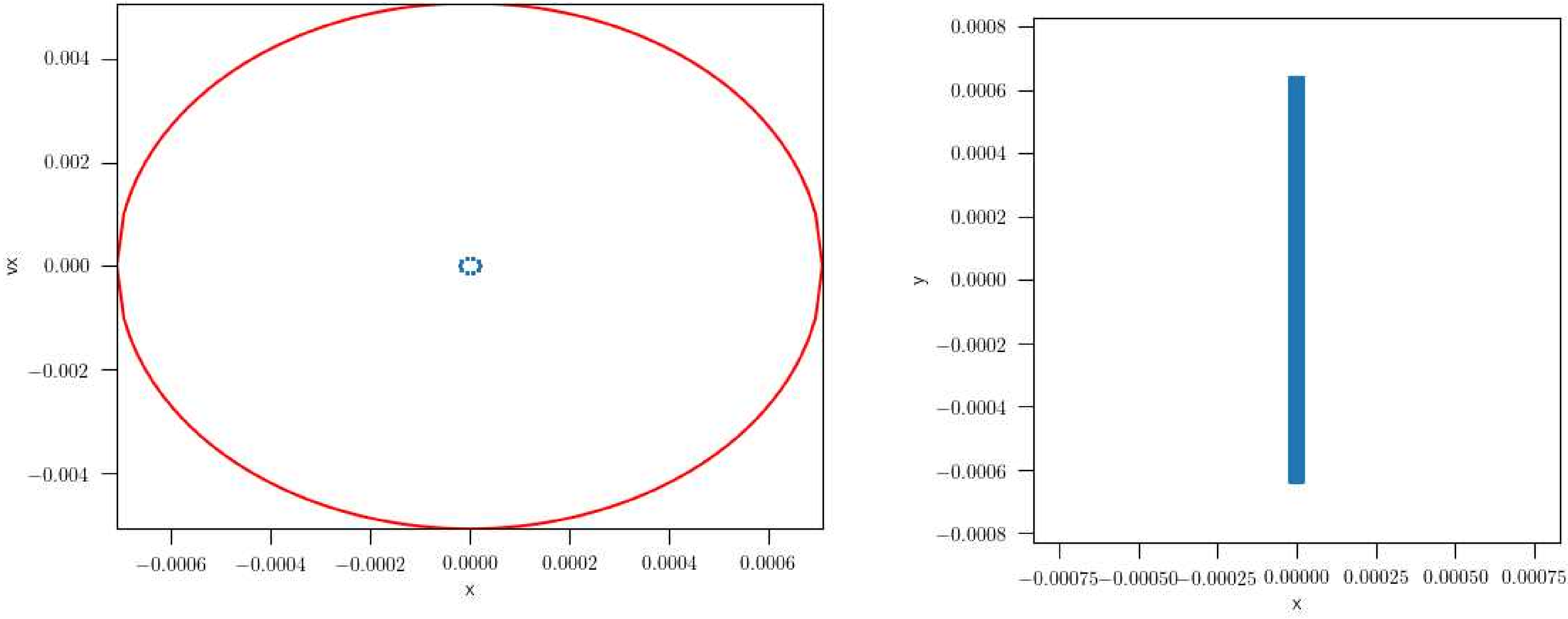
$$q = 0.9$$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.9661   --norbits 50
```

small x, Y-elongated orbits (box orbit)

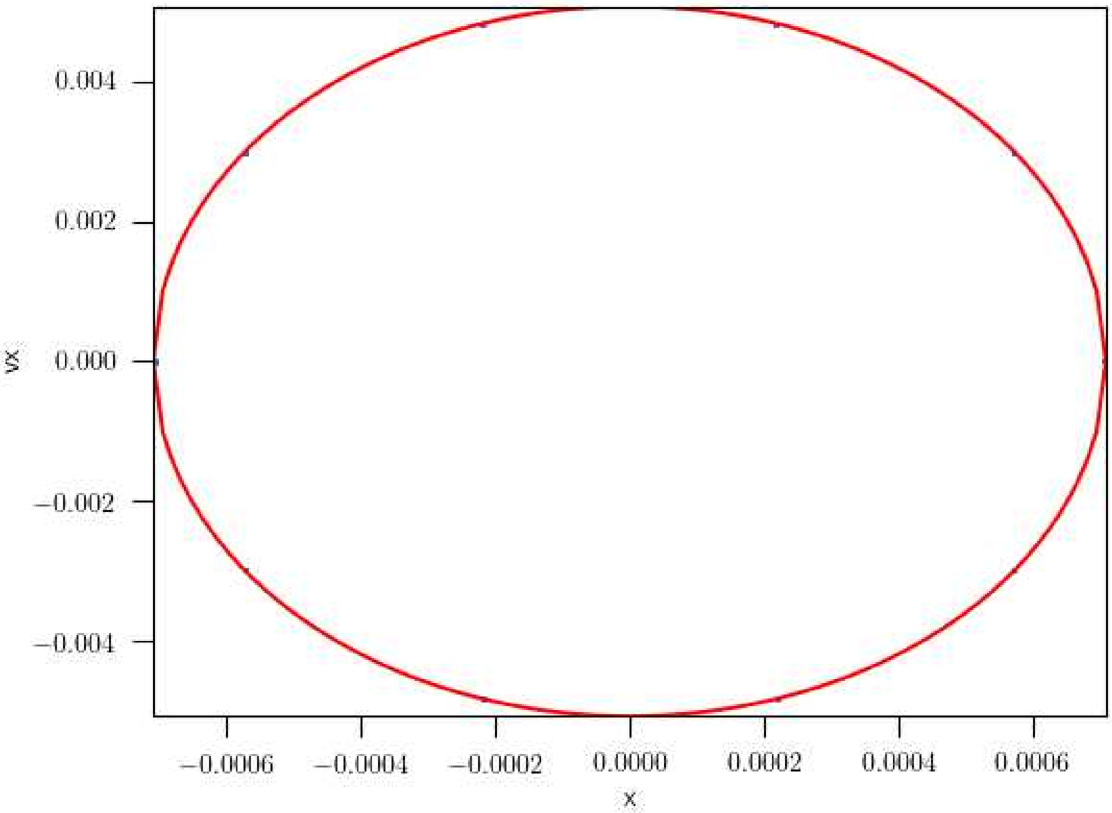
$$q = 0.9$$

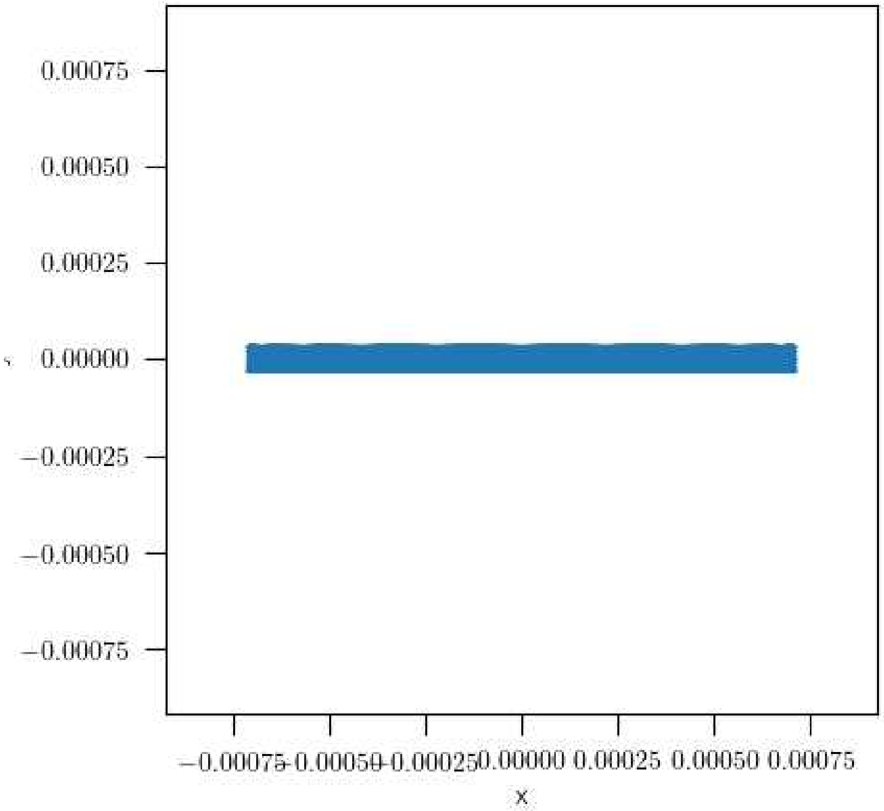



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.9661   --x 0.00002
```

large x, X-elongated orbits (box orbit)

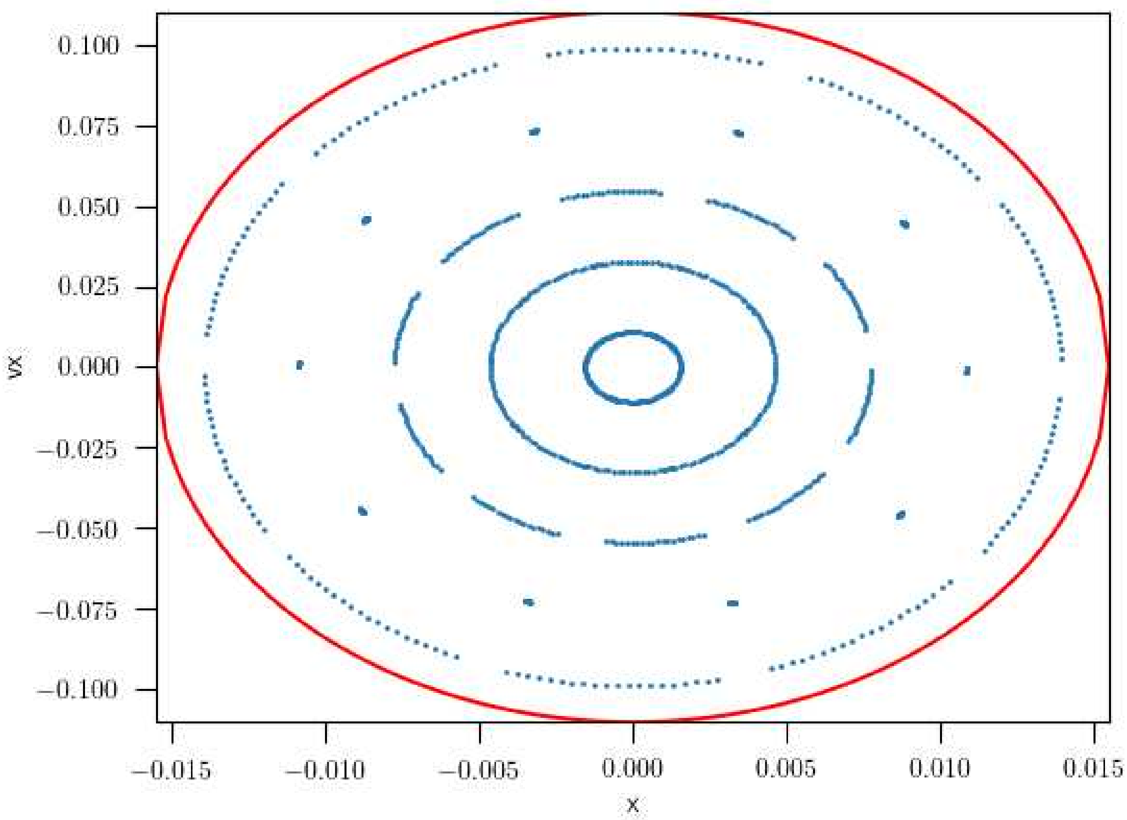
$$q = 0.9$$





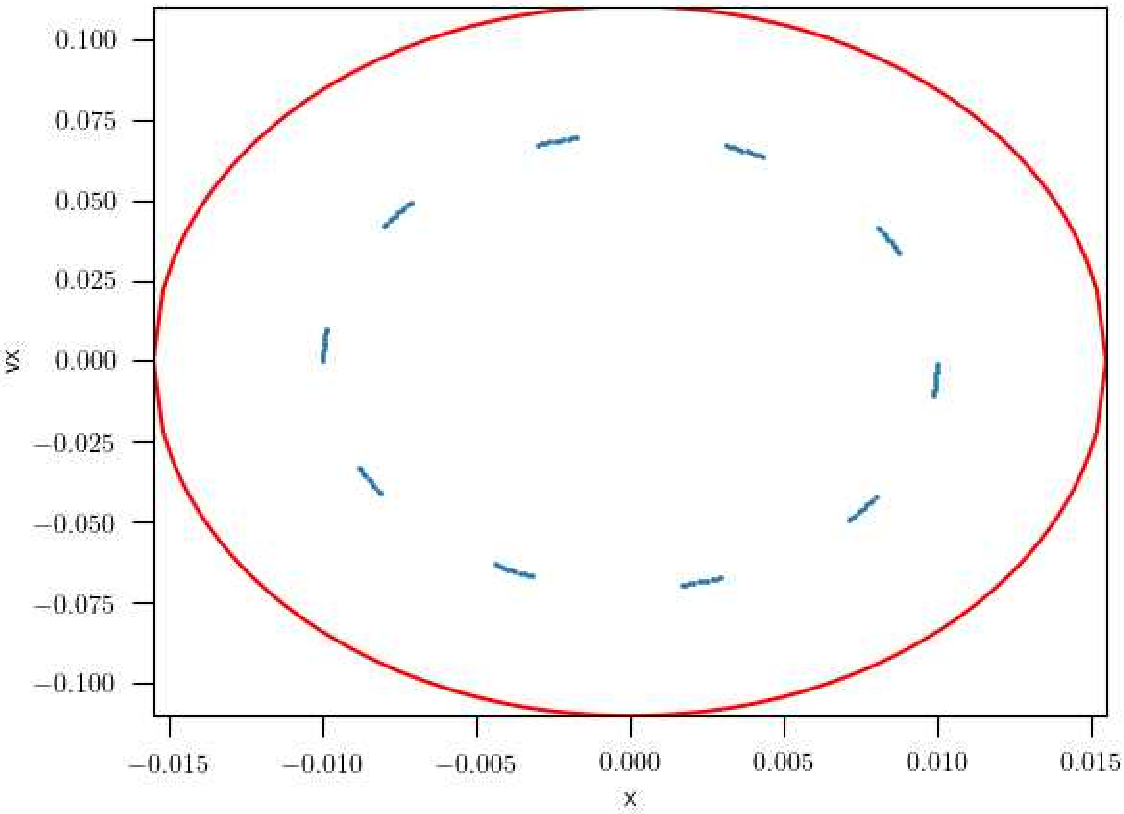

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.9661   --x 000709
```

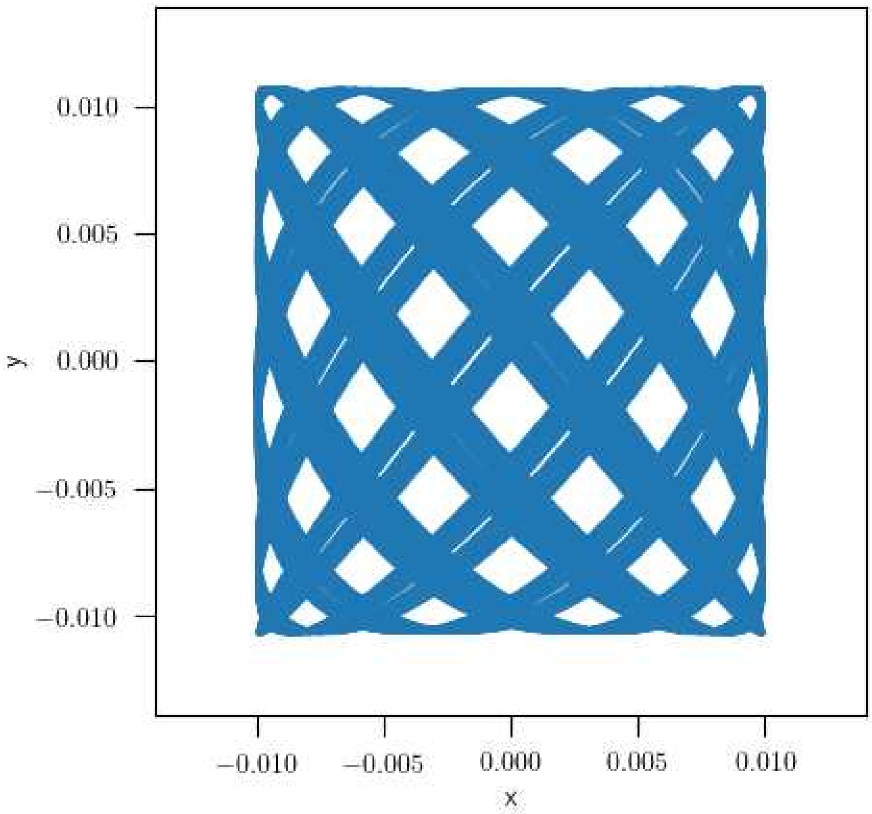
Increasing energy : perturbed harmonic oscillator (coupling terms)




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96
```

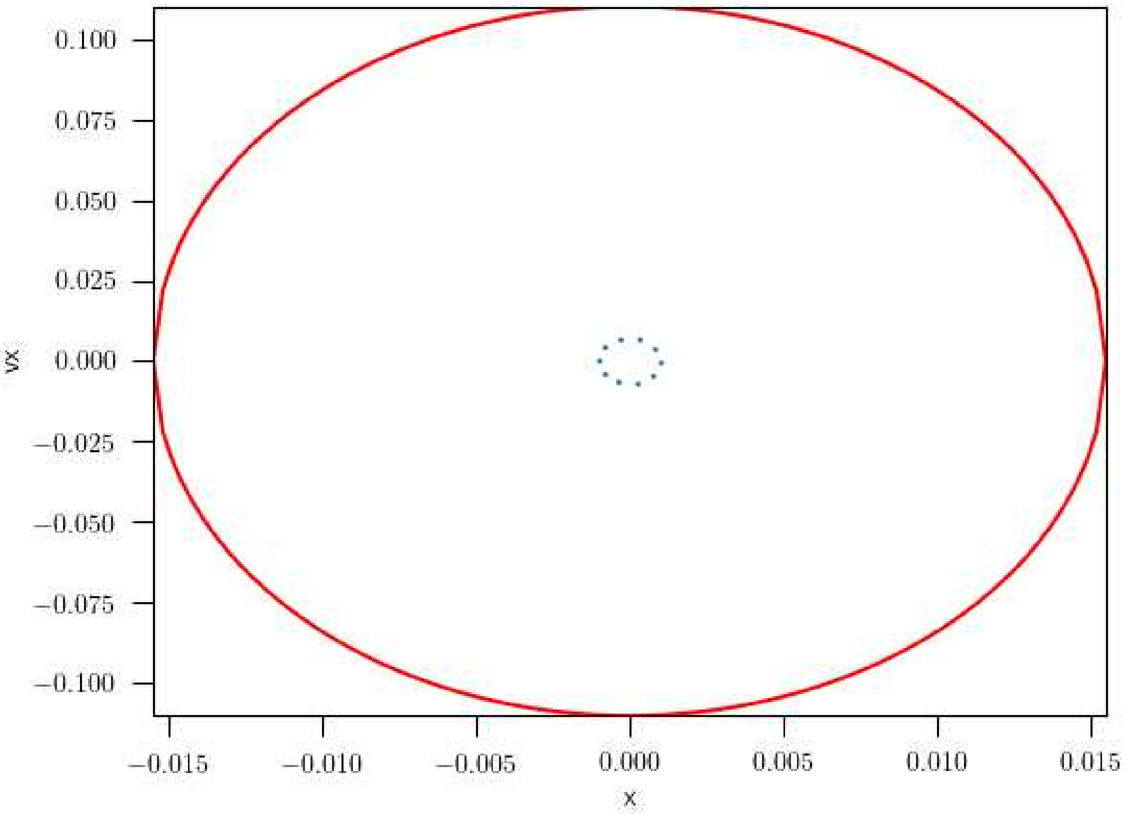
Increasing energy : perturbed harmonic oscillator

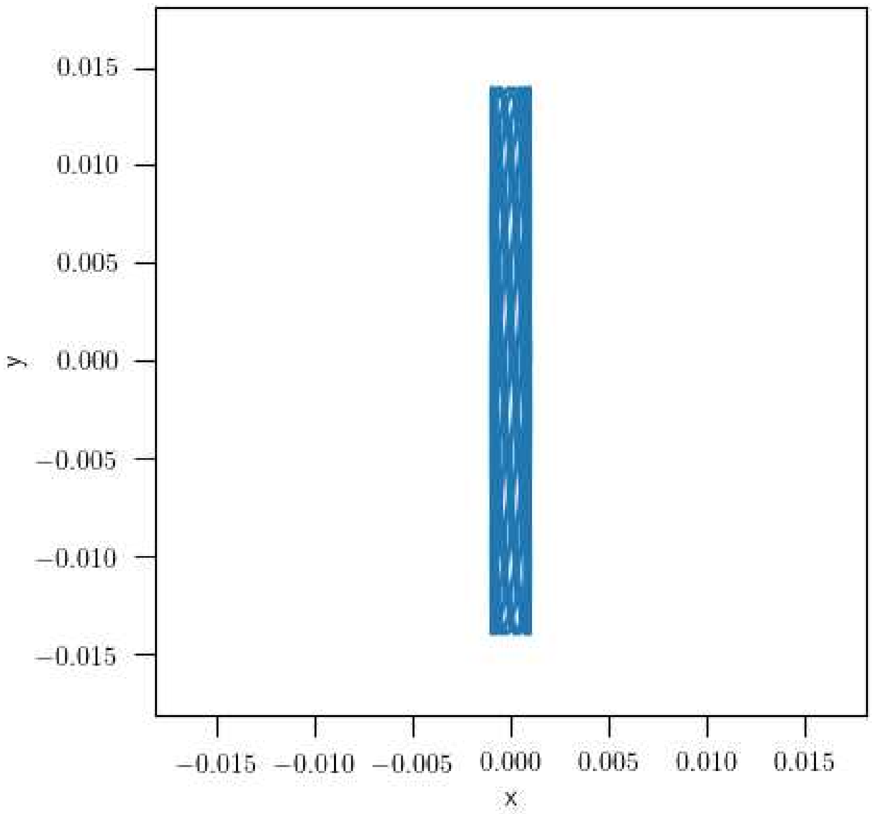





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96   --x 0.01
```

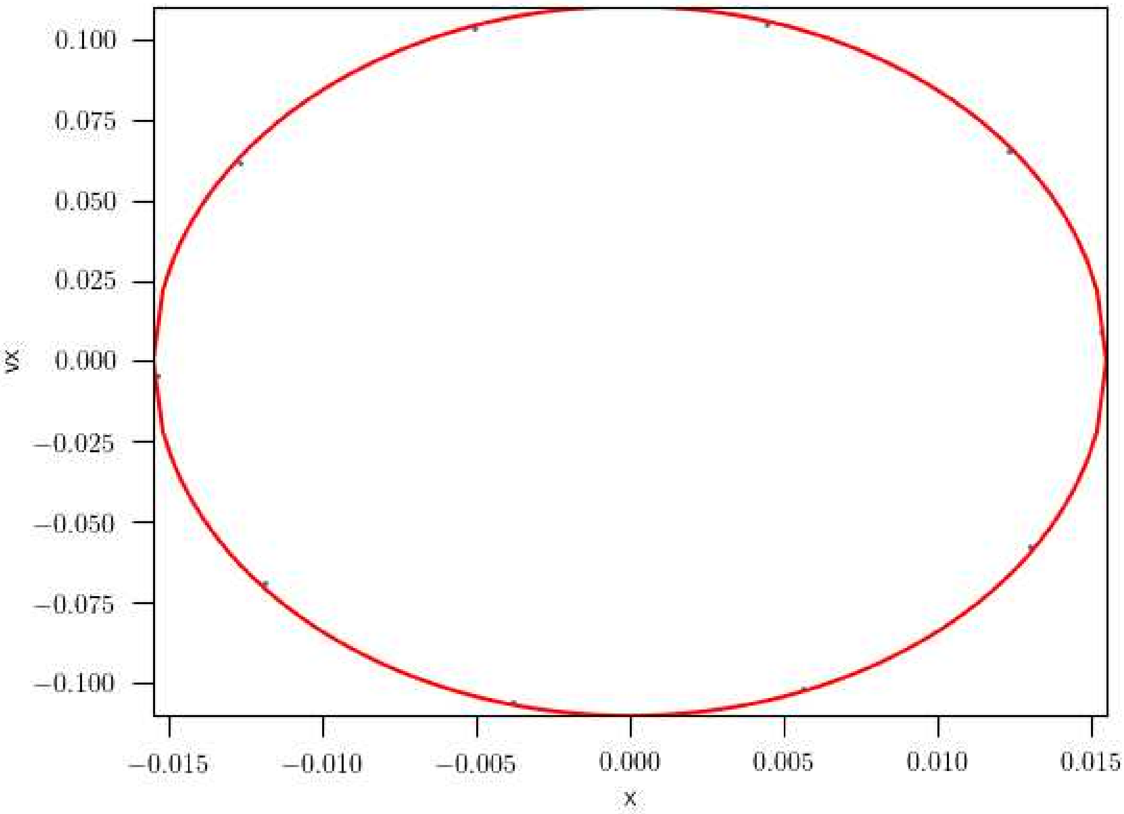
small x, Y-elongated orbits (box orbit)

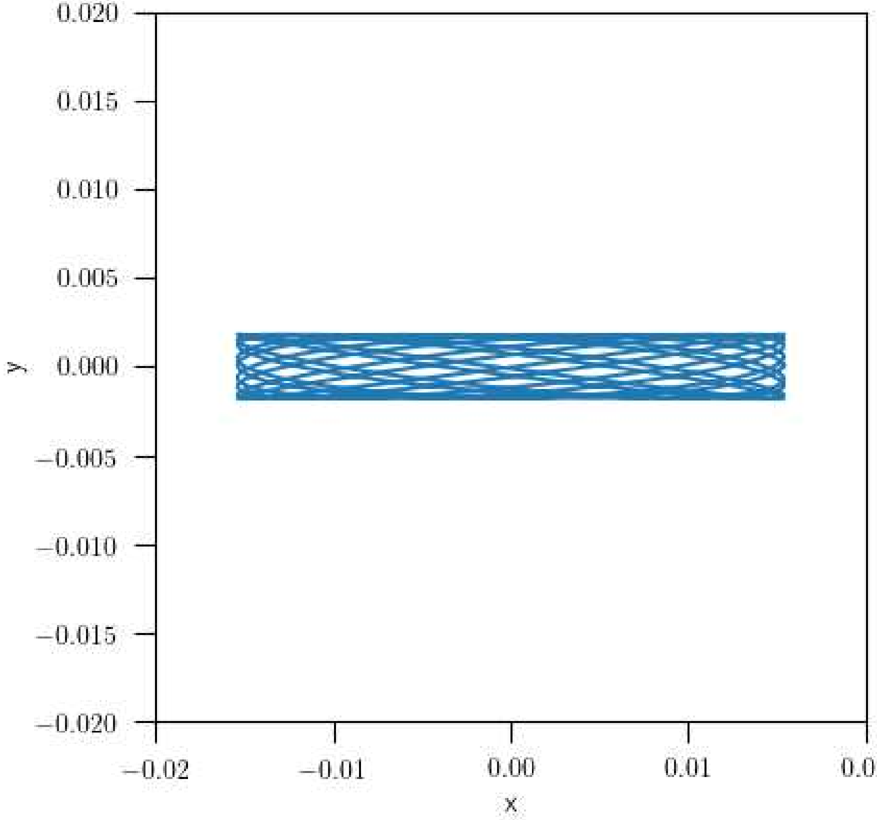





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96   --x 0.001  --nlaps 10
```

large x, X-elongated orbits (box orbit)






```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.96   --x 0.0154 --nlaps 10
```

$$R \gg R_c$$

Motions for $R \gg R_c$

$$\phi(x,y) = \frac{1}{2} V_0^2 \ln\left(R_c + x^2 + \frac{y^2}{q^2}\right)$$

$$\simeq \frac{1}{2} V_0^2 \ln\left(x^2 + \frac{y^2}{q^2}\right) \underset{q \simeq 1}{\sim} \frac{1}{2} V_0^2 \ln\left(R^2\right)$$

Orbit families

① **box orbits** (disturbed 2D harmonic oscillator)

$V_{//}$: $v_y \simeq 0$

if $v_y = 0$: radial orbit ($L_z = 0$)

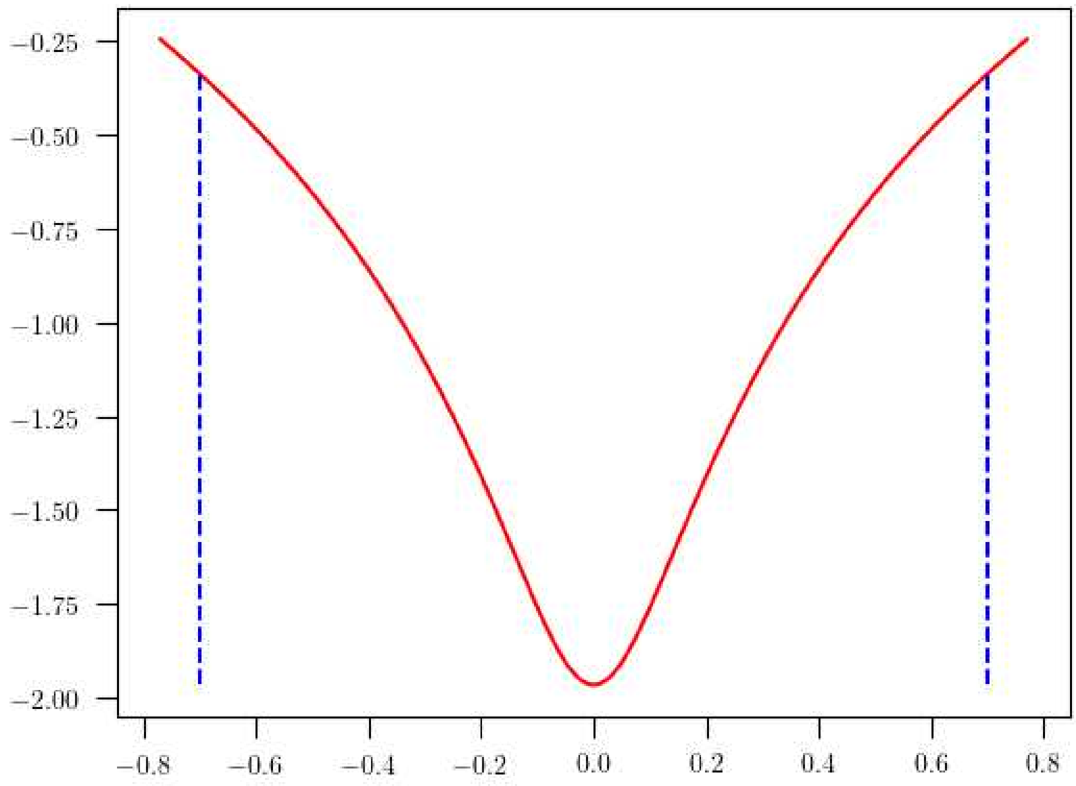
② **loop orbits**

$V_\perp$: $v_y \simeq V_0$

if $v_y = V_0$: circular orbit

$q = 1$

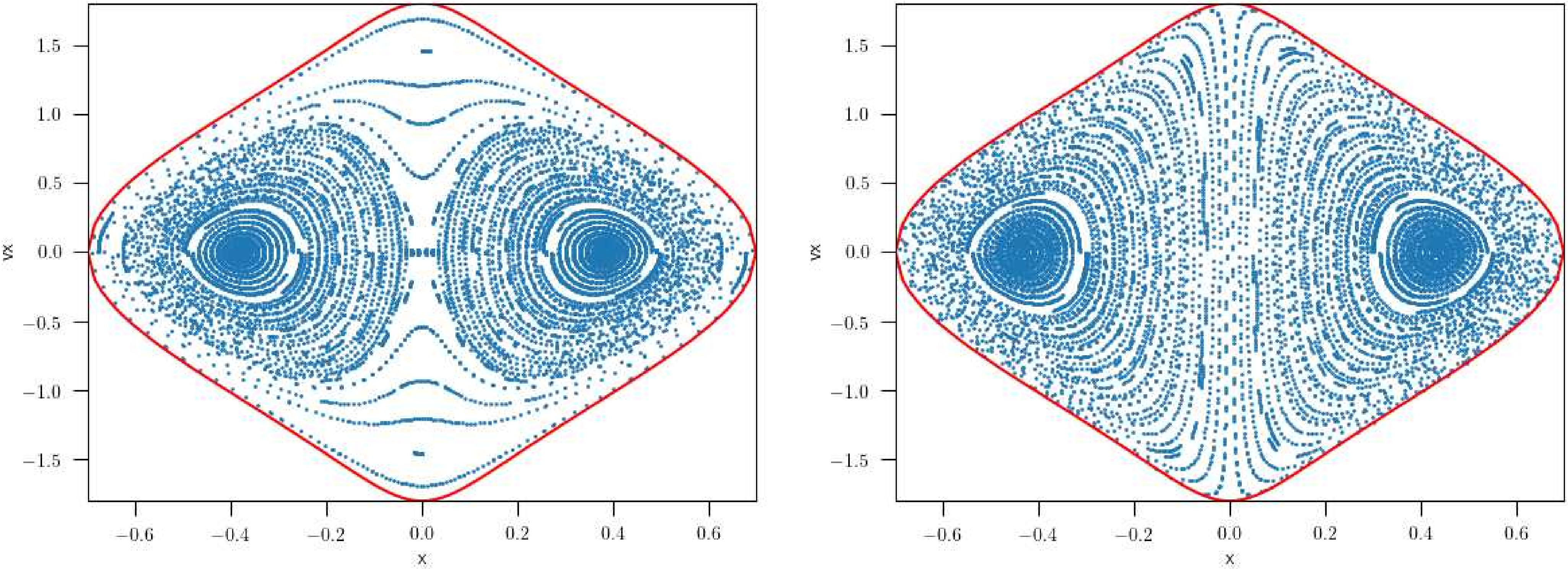
Potential and energy




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --plotpotential
```

Phase space

$q = 0.9$ $\qquad$ $q = 1.0$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337 --norbits 100
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```
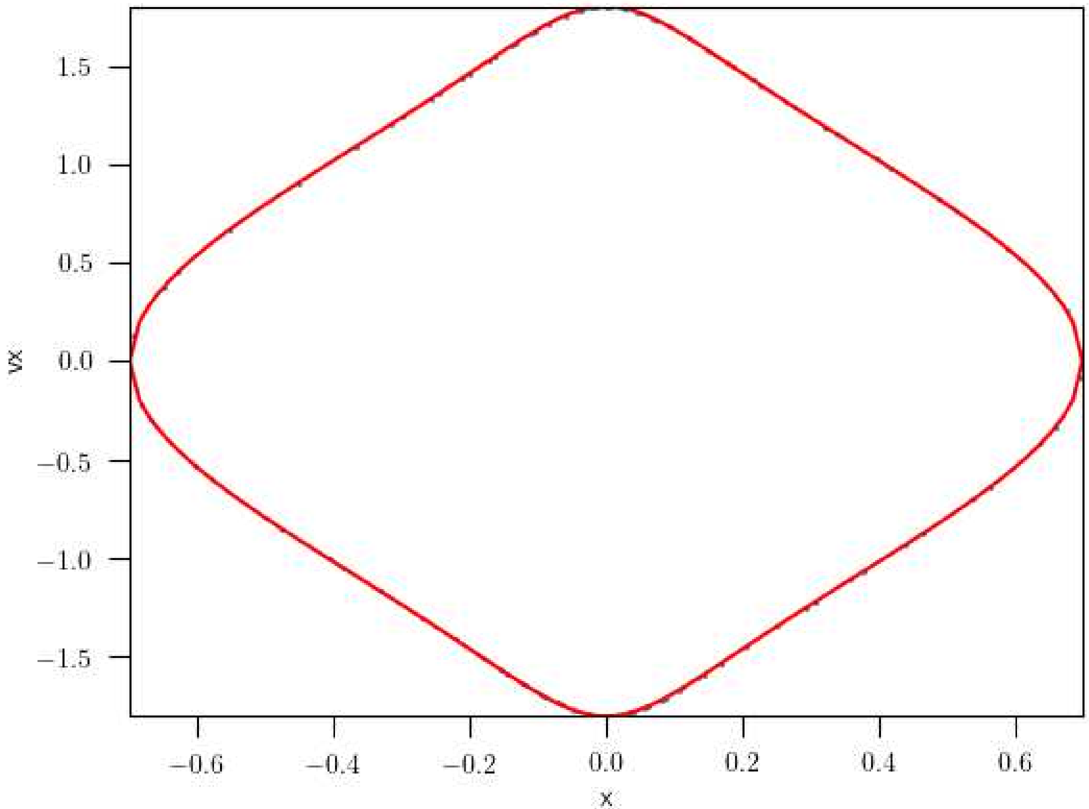
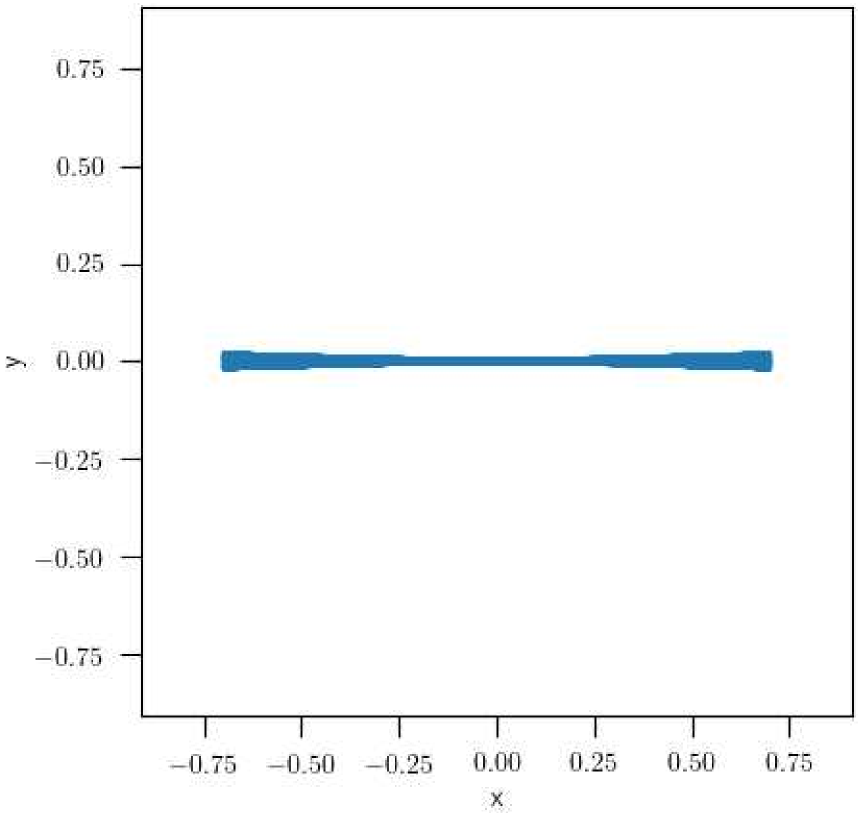

Box orbits






```
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```
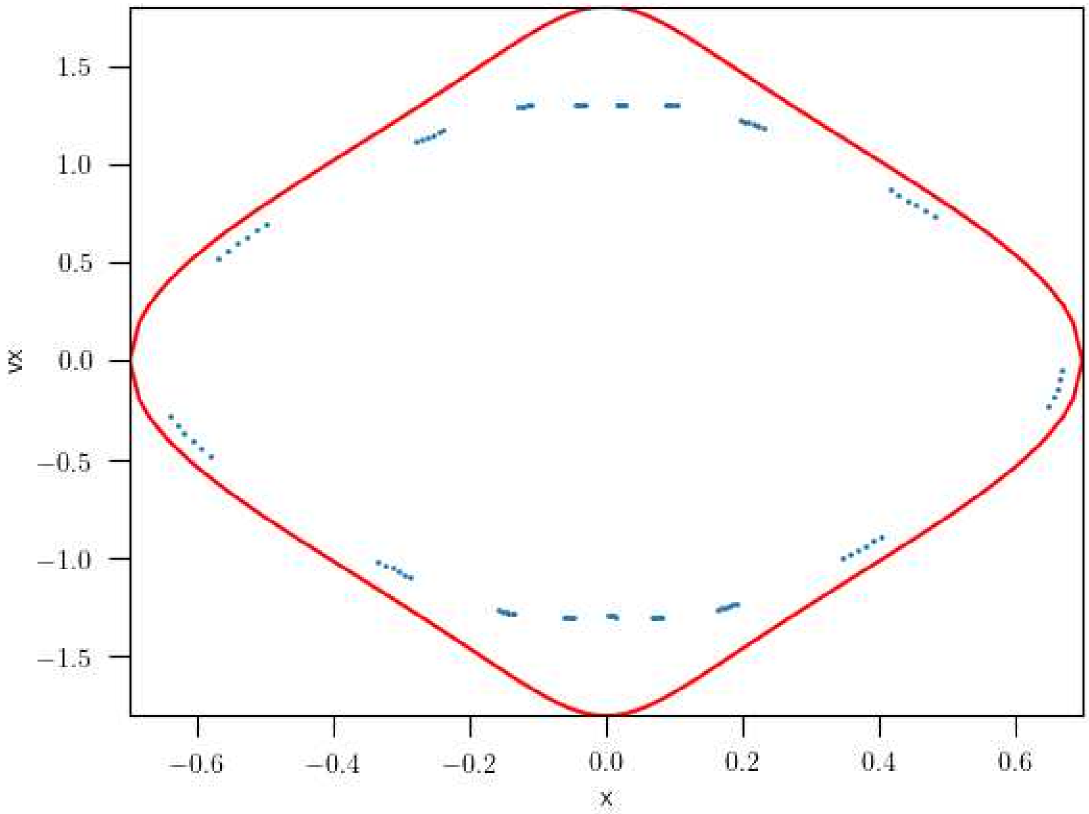
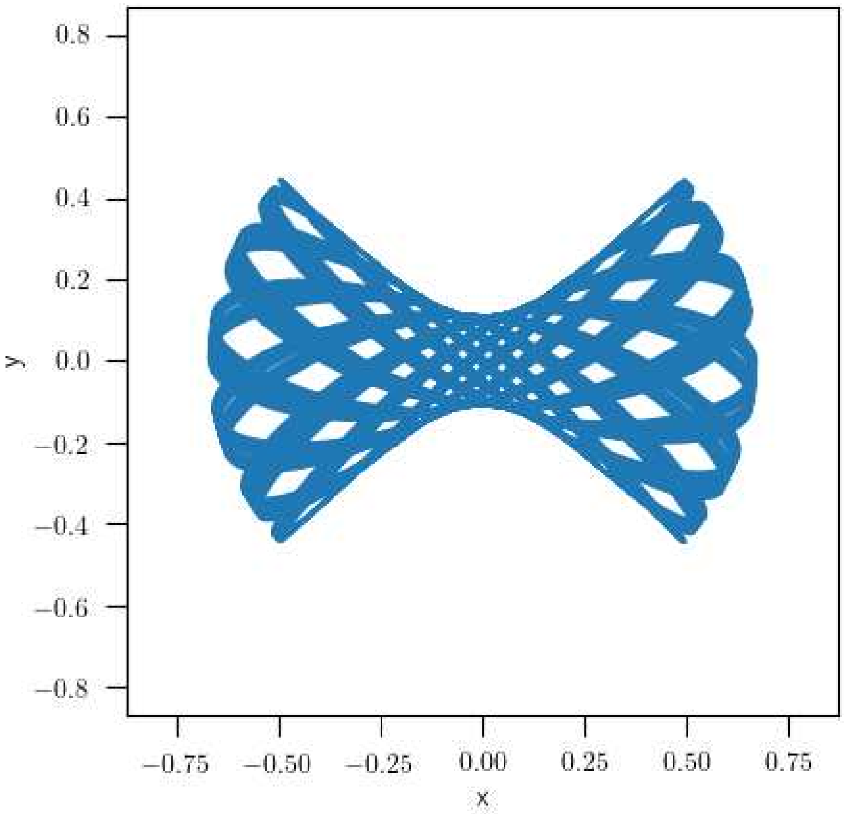

Box orbits






```
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```
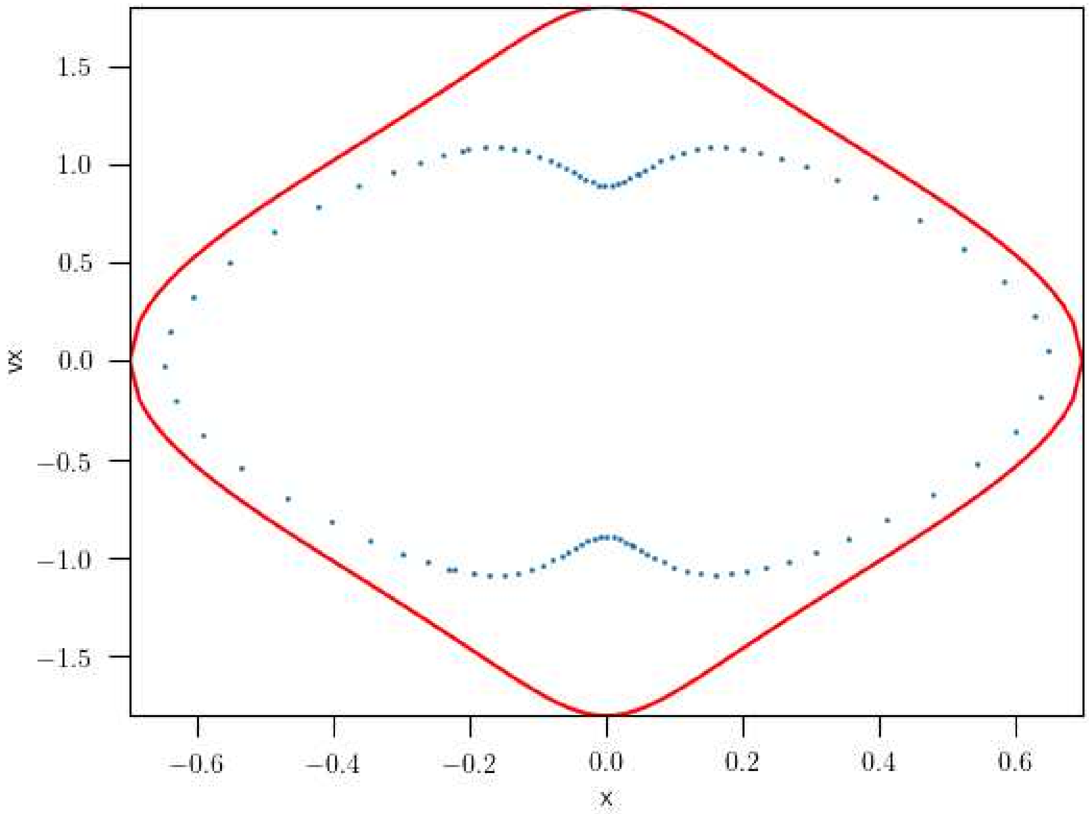
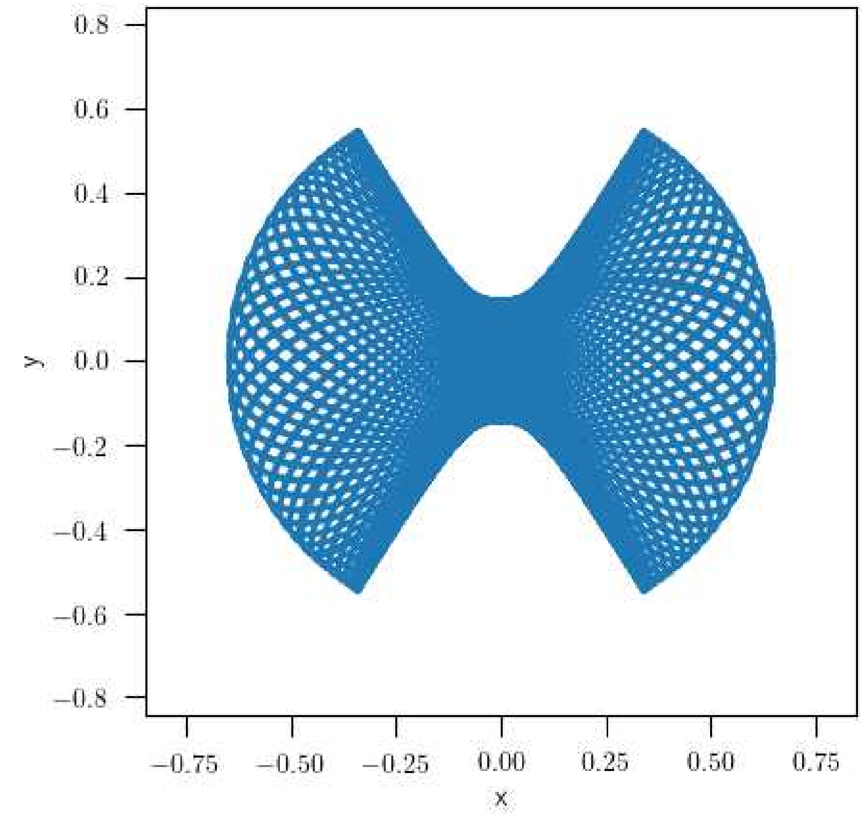

Box orbits






```
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```
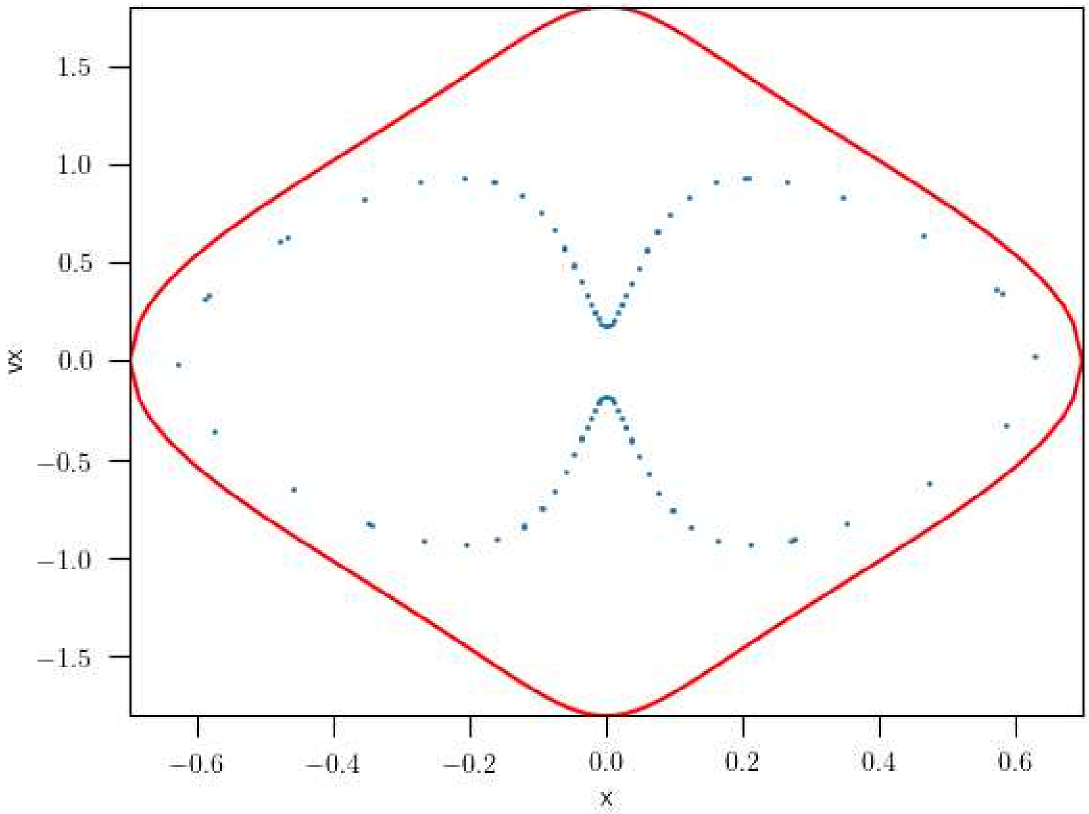
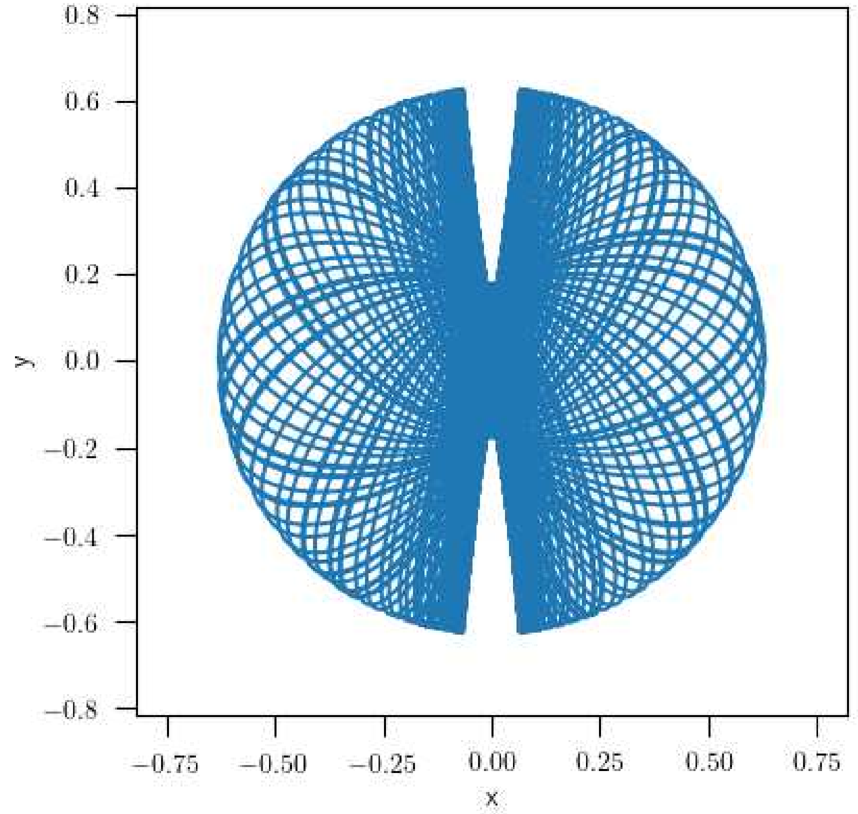
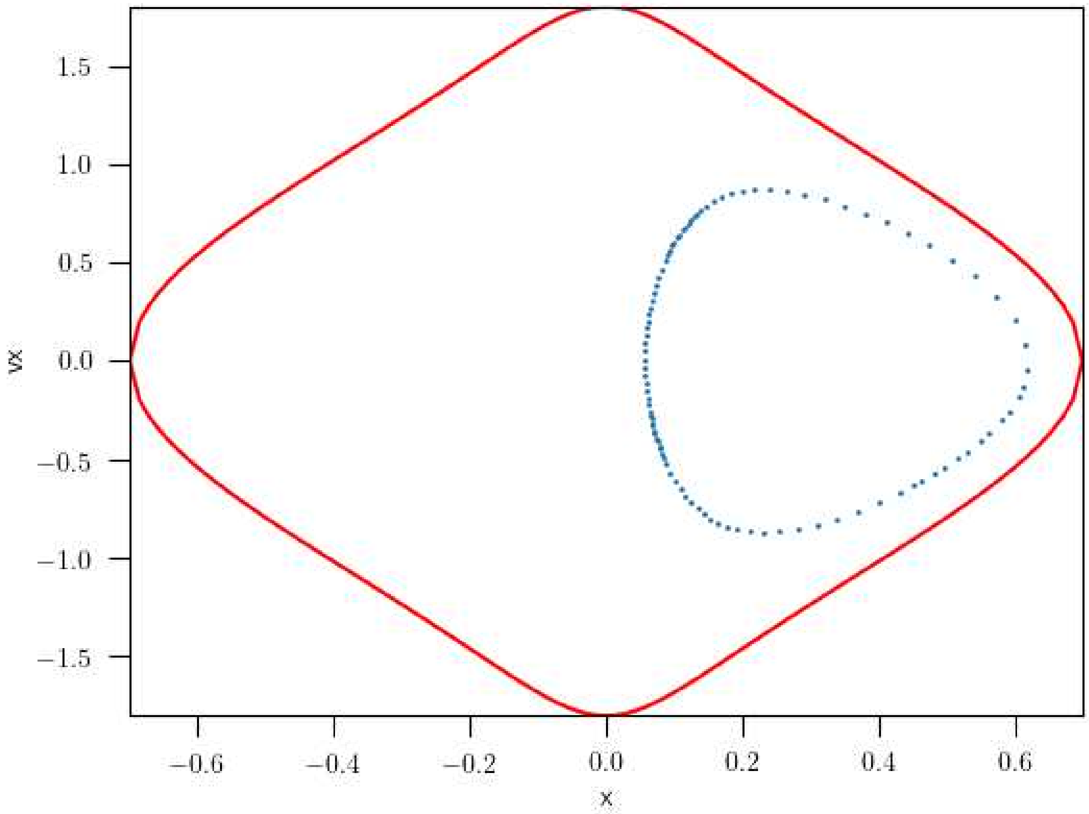

Box orbits

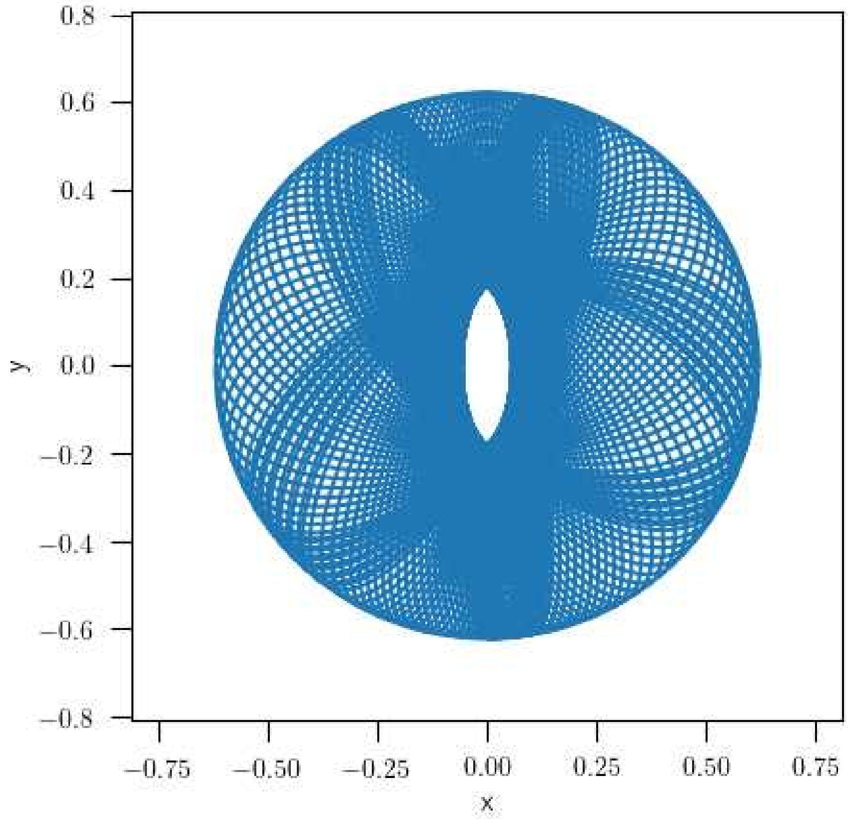





```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.63
```

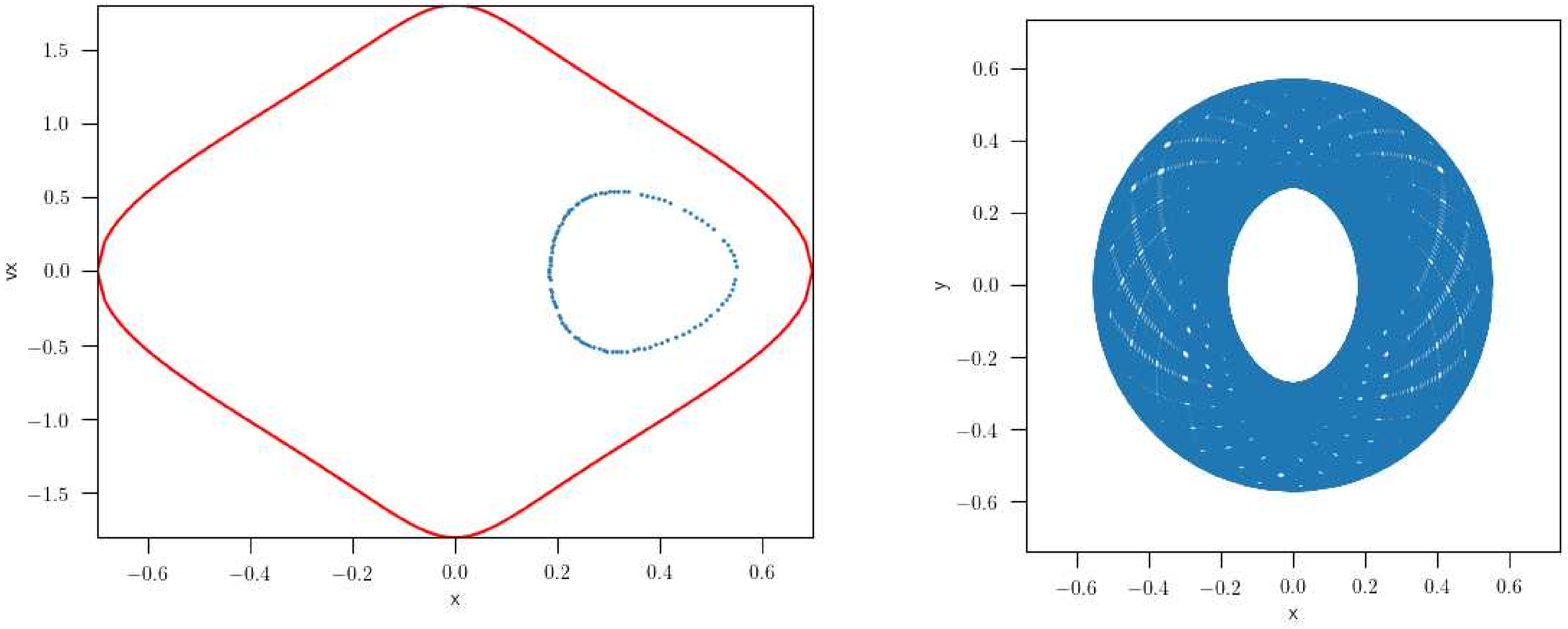
Loop orbits






```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.62
```

Loop orbits




```
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```
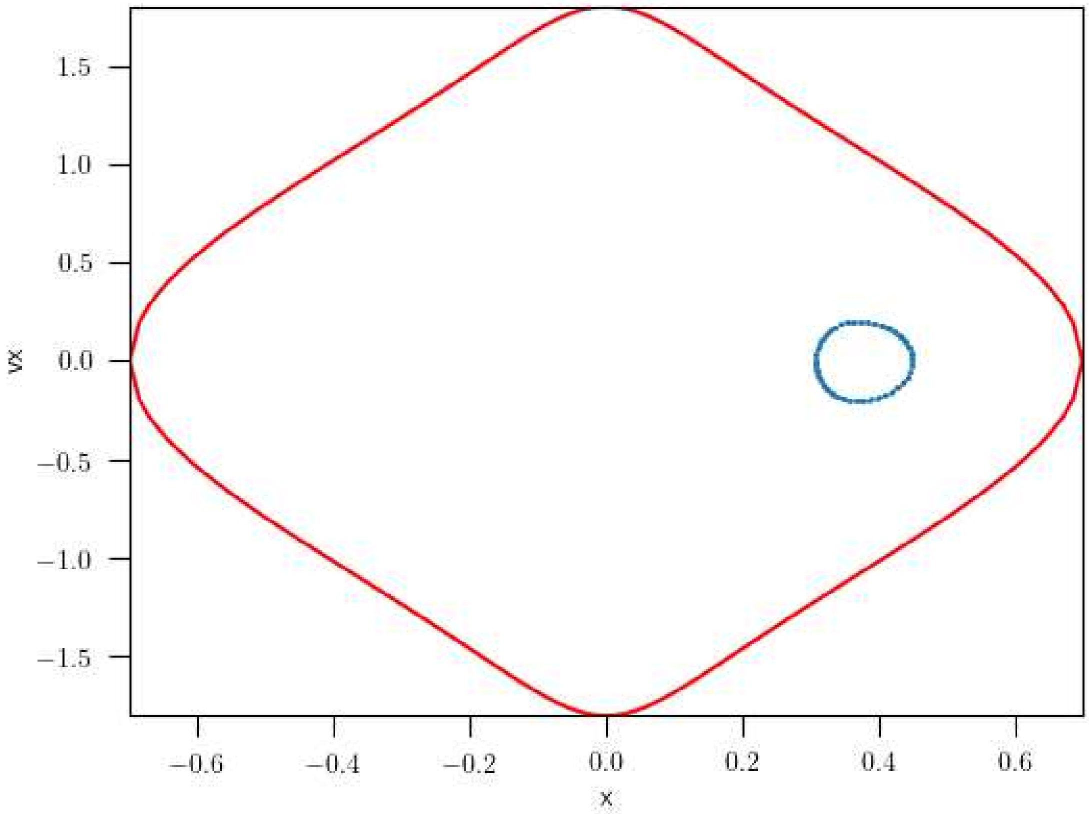
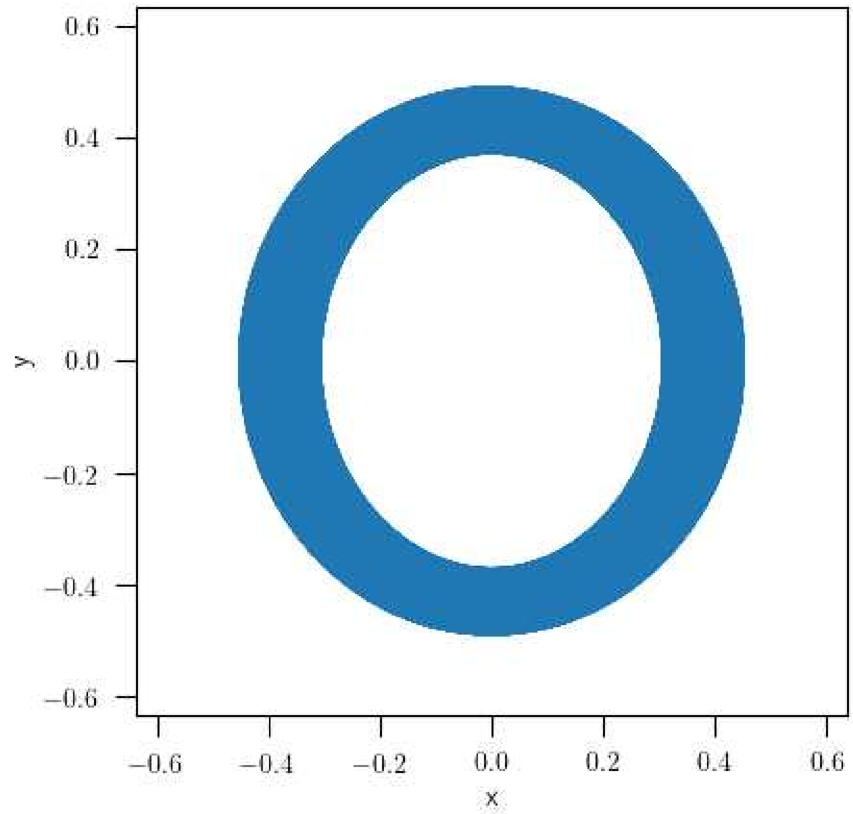

Loop orbits






```
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```
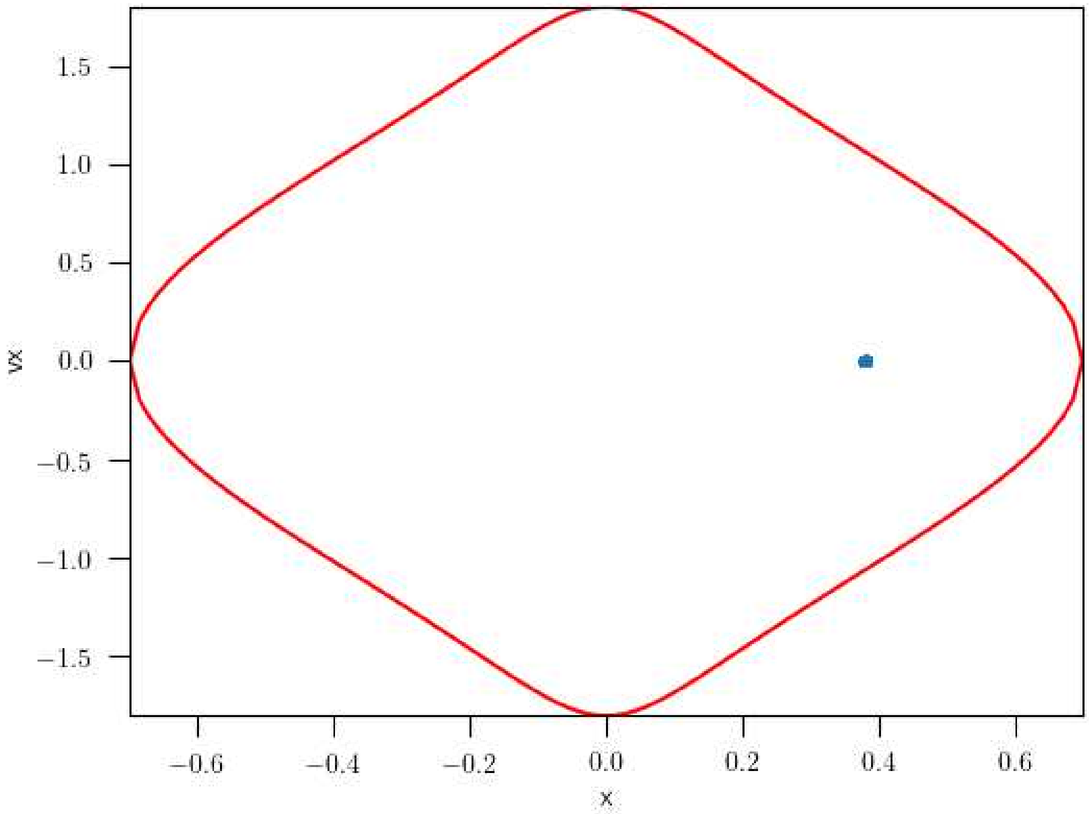
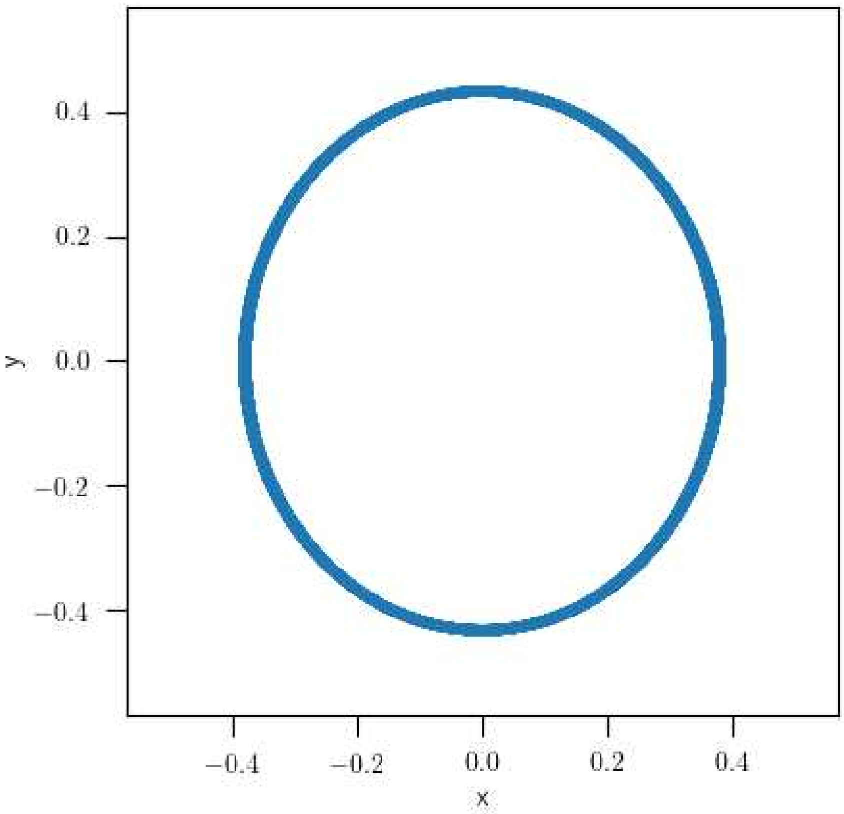

Loop orbits

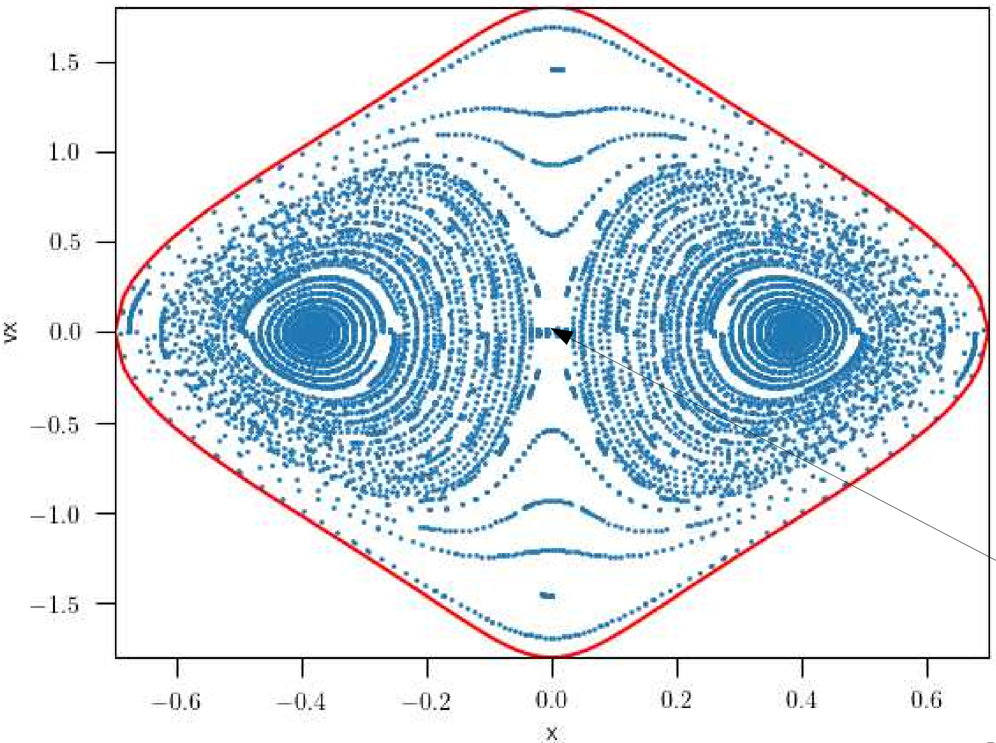


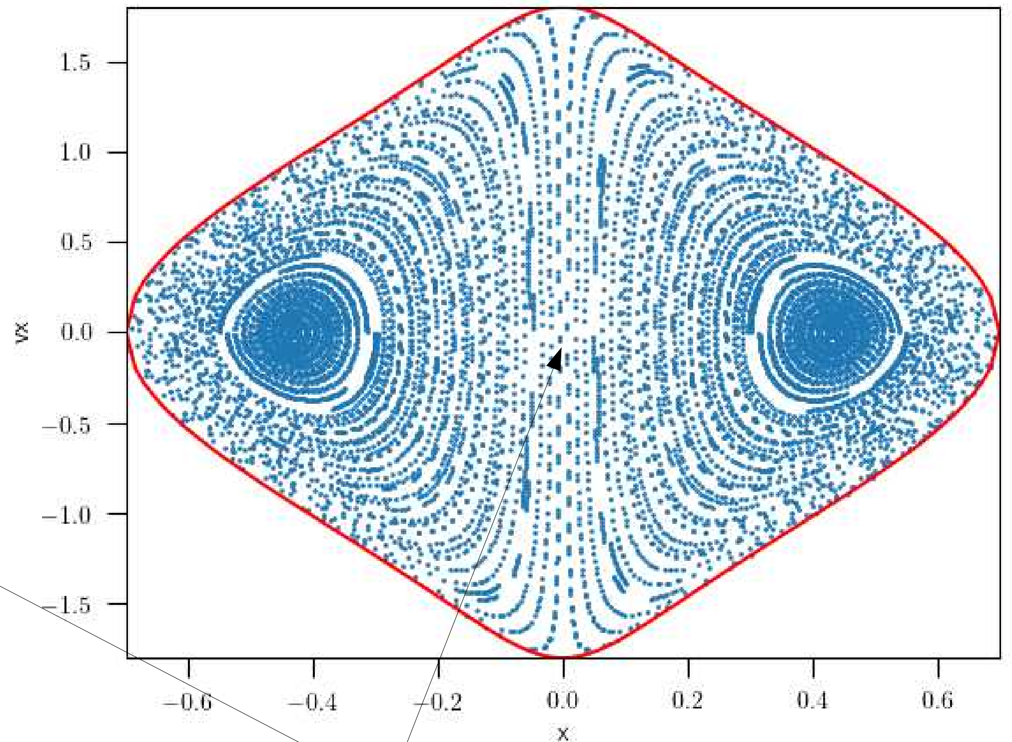



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9    -E -0.337 --x 0.374
```

Box orbits elongated towards the y axis

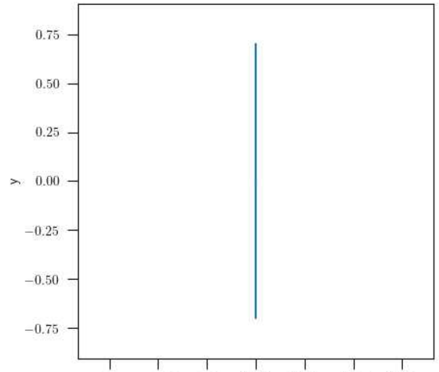
$q = 0.9$ $q = 1.0$





unstable

stable

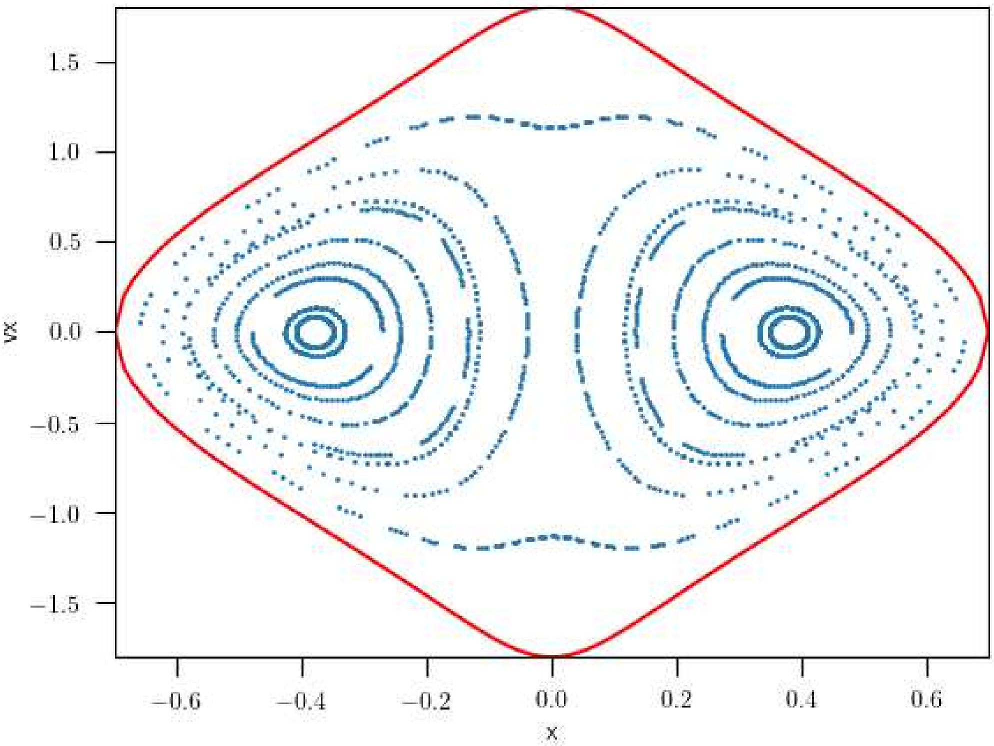



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337 --norbits 100
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337 --norbits 100
```

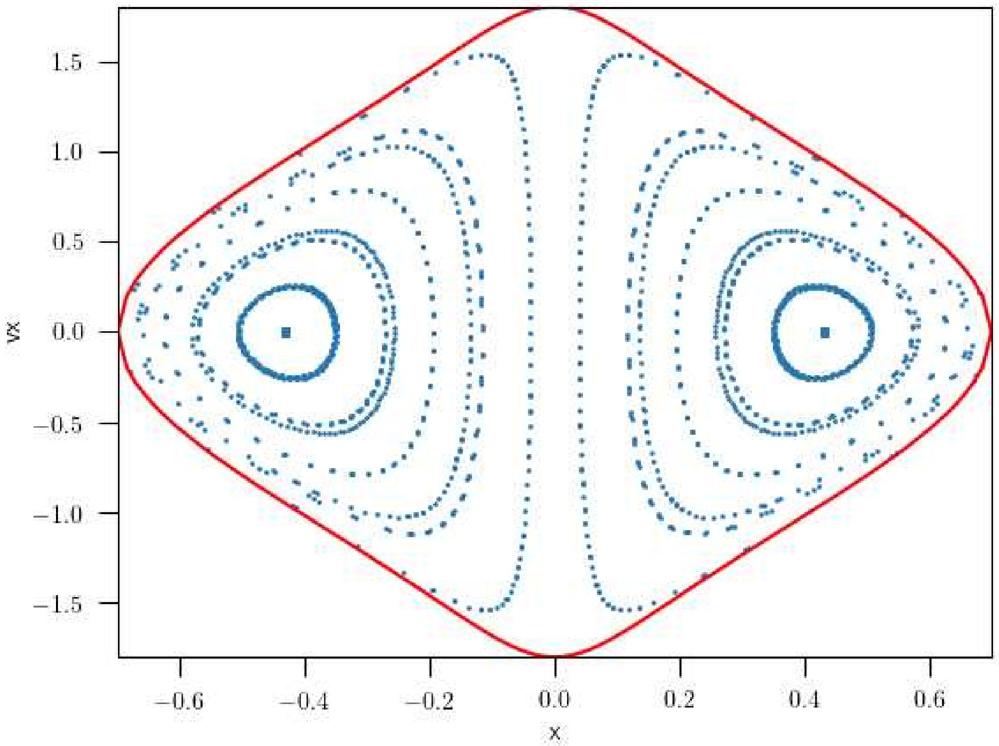
Integral of motions ?

Integral of motions ?

$q = 0.9$



$q = 1.0$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --norbits 18
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --norbits 18
```

Integrals of motions

① "nearly circular orbits"

Angular momentum conservation $\qquad L_z = x\dot y - y\dot x$

can we compute $x = x(\dot x)$ in the plane $y = 0$?

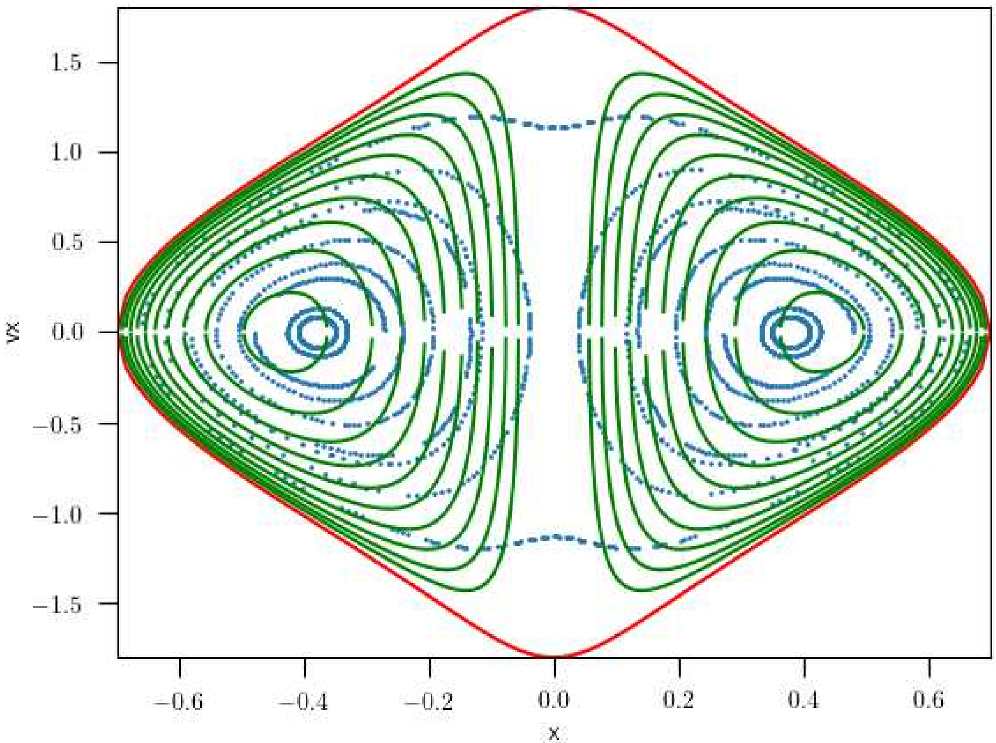
$$L_z = x\dot y$$

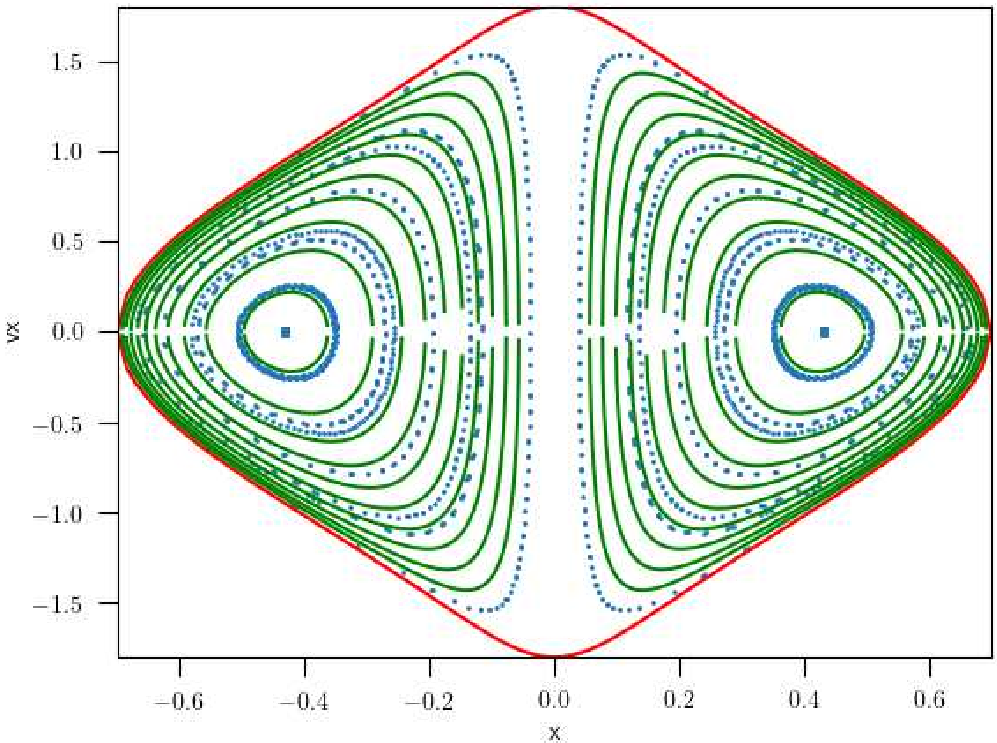
$$E = \frac{1}{2}\dot x^2 + \frac{1}{2}\dot y^2 + \phi(x, y=0)$$

$$\boxed{\dot x = \sqrt{2(E-\phi) - \dot y^2} = \sqrt{2\left(E - \phi - \frac{L_z^2}{x^2}\right)}}$$

L_z

$q = 0.9$ $q = 1.0$






```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --norbits 18  --add_ILz
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --norbits 18  --add_ILz
```

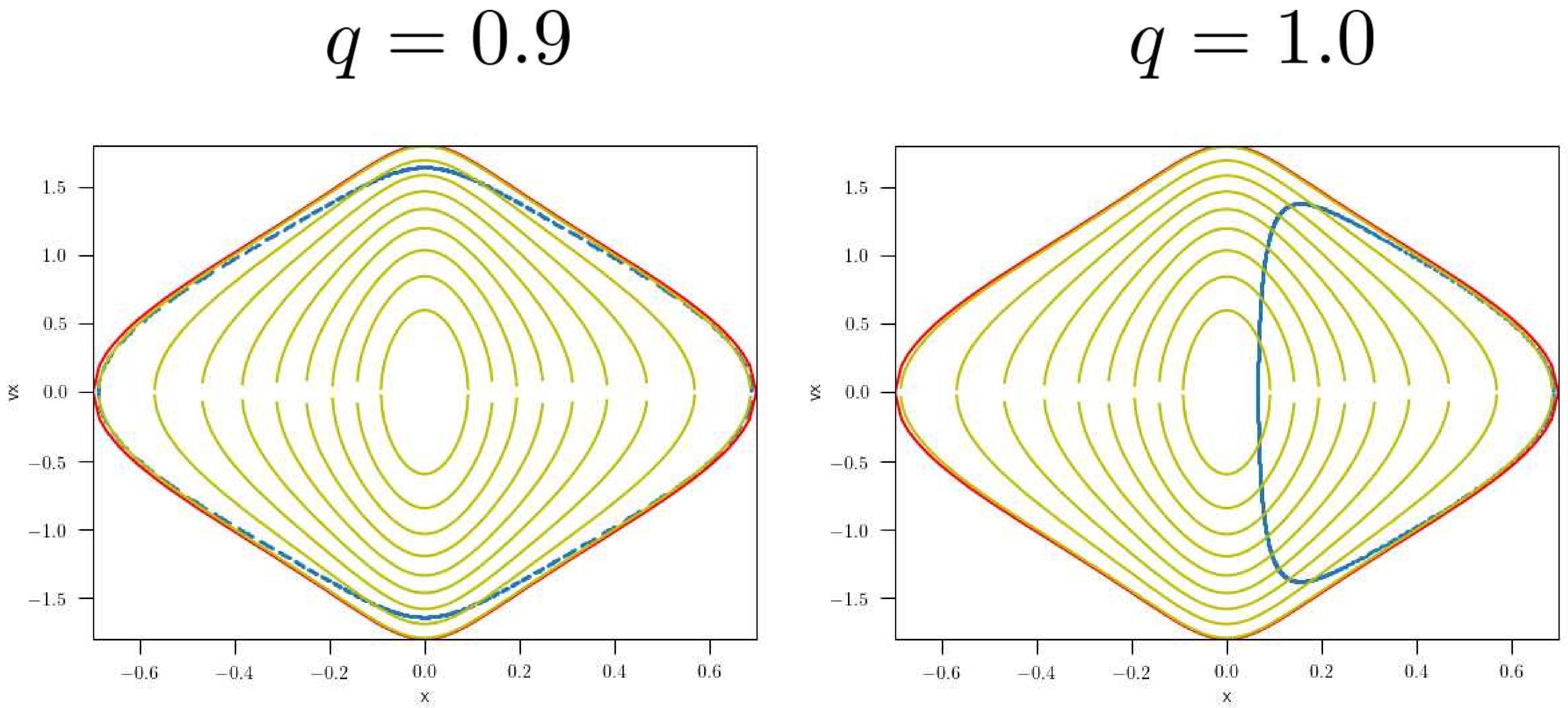
Integrals of motions

② Motion parallel to the long axis $\quad (y = \dot{y} = 0)$

$$H_x = \frac{1}{2}\dot{x}^2 + \phi(x, y=0) = E_x \quad \text{(harmonic oscillator)}$$

$$\boxed{\dot{x} = \sqrt{2\left(E_x - \phi(x, y=0)\right)}}$$

H_x

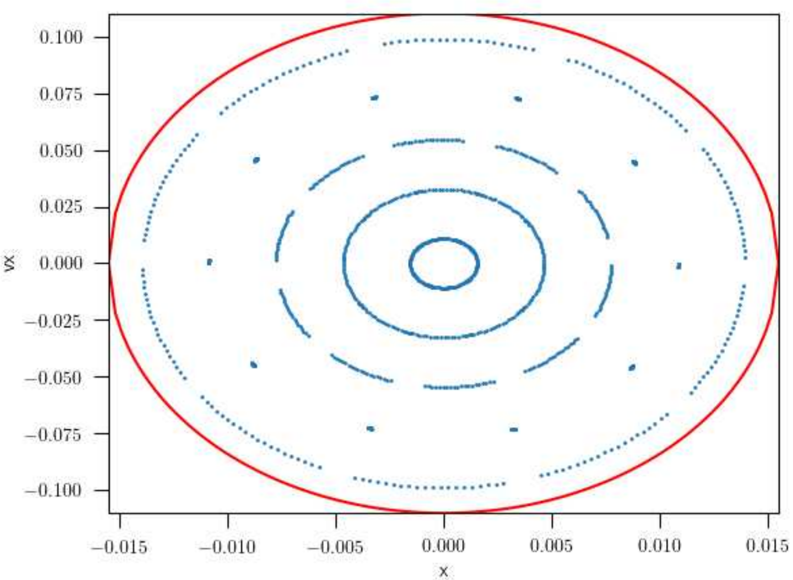


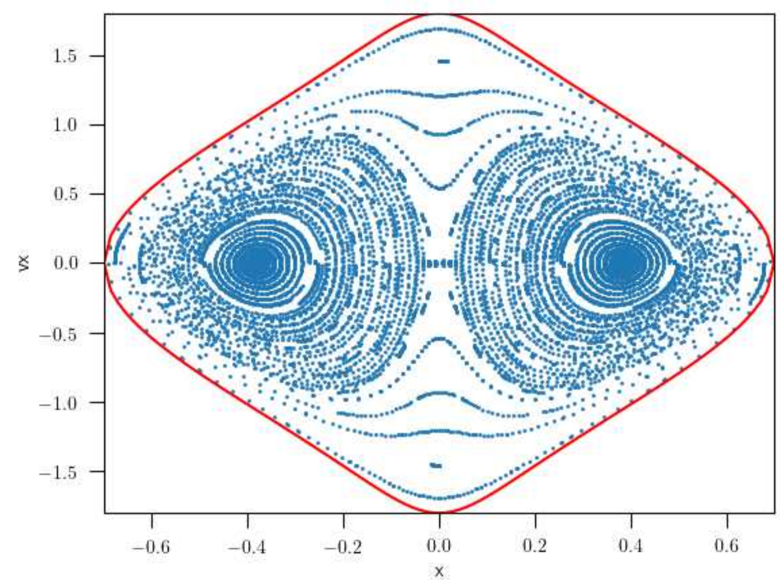

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --x 0.69  --nlaps 1000    --add_Ix
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --x 0.69  --nlaps 1000    --add_Ix
```

$$R \sim R_c$$

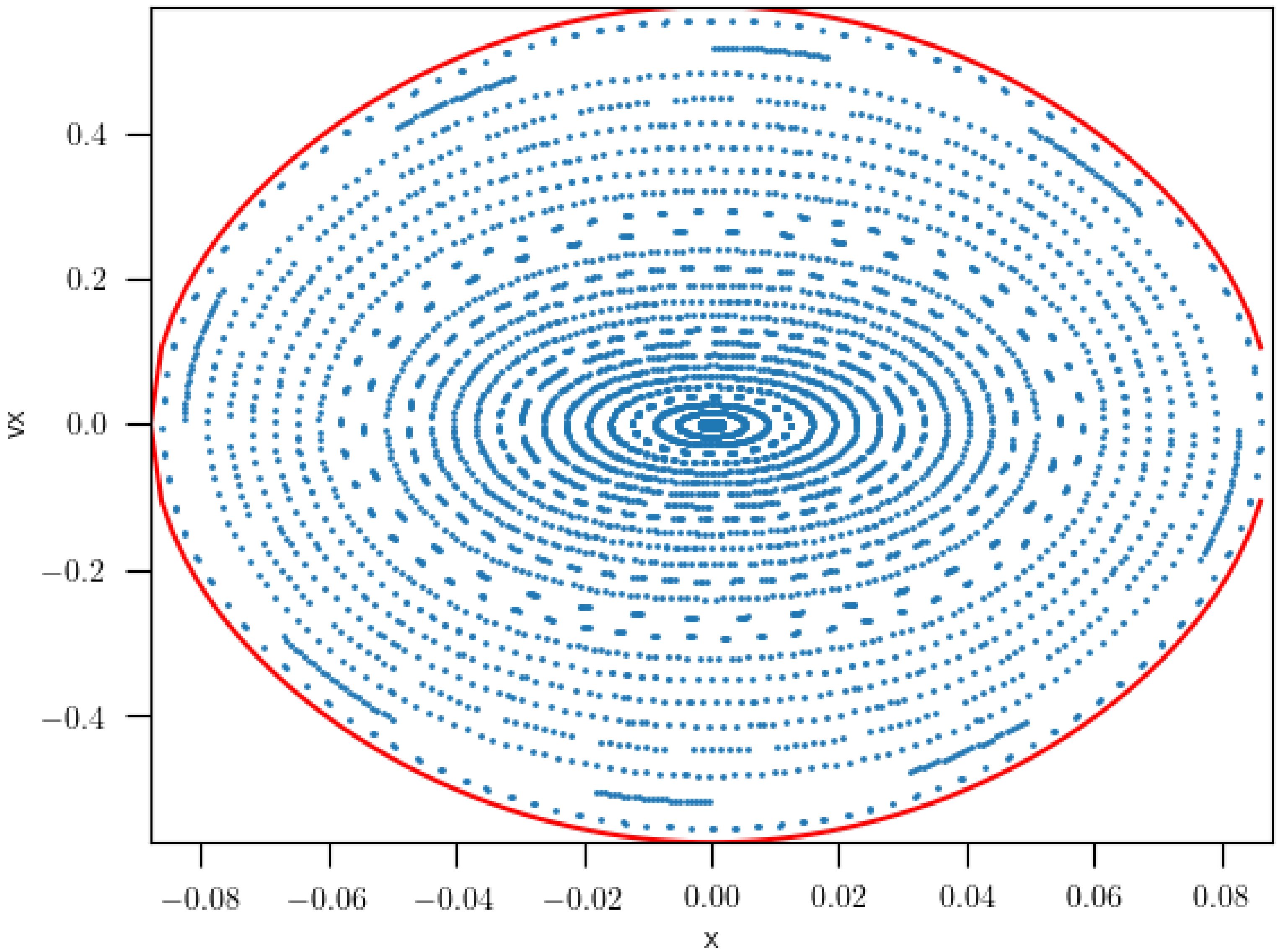
Family decoupling

from low energy 1 family | to | high energy 2 famillies



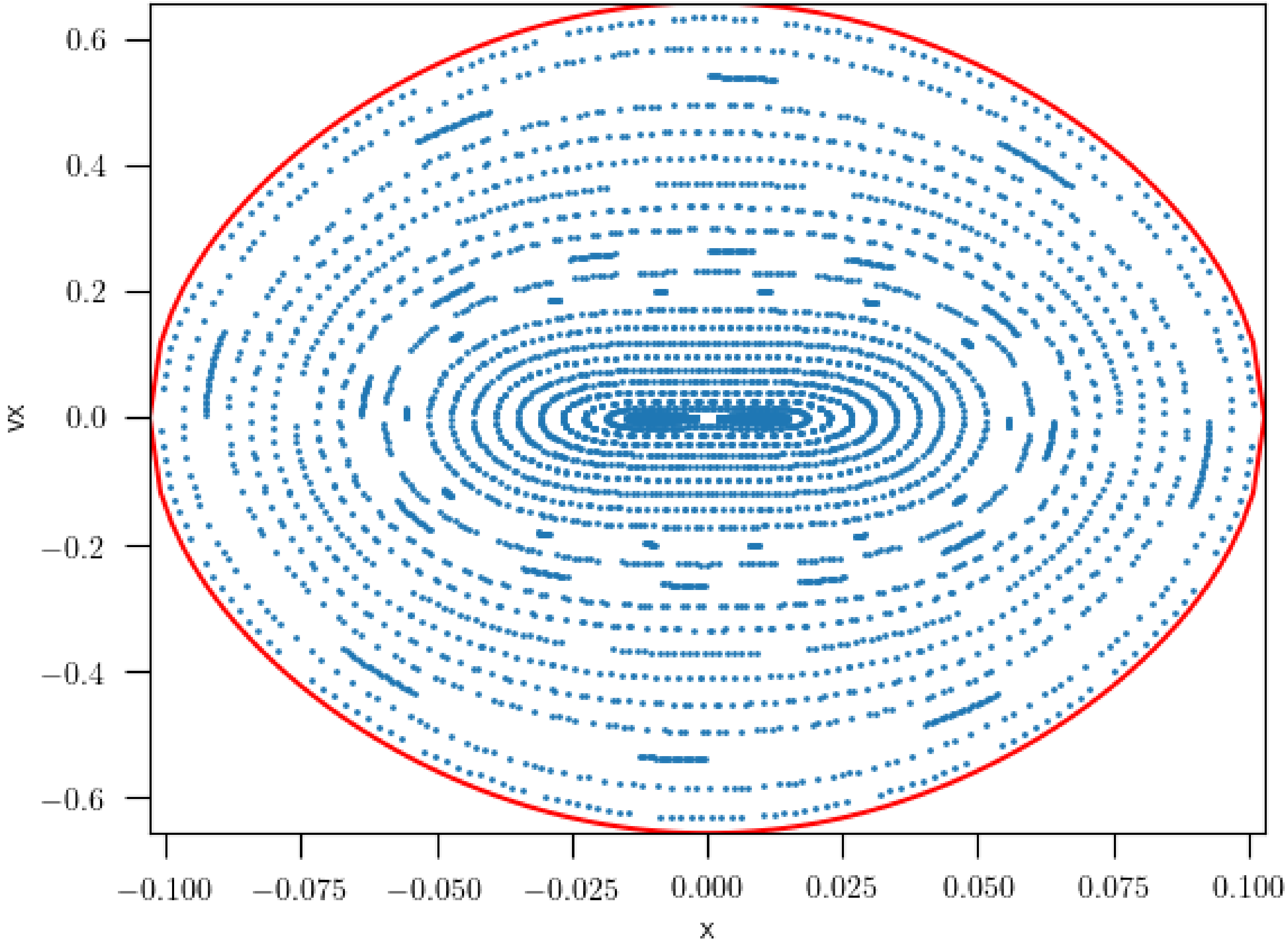


$$E = -1.8$$



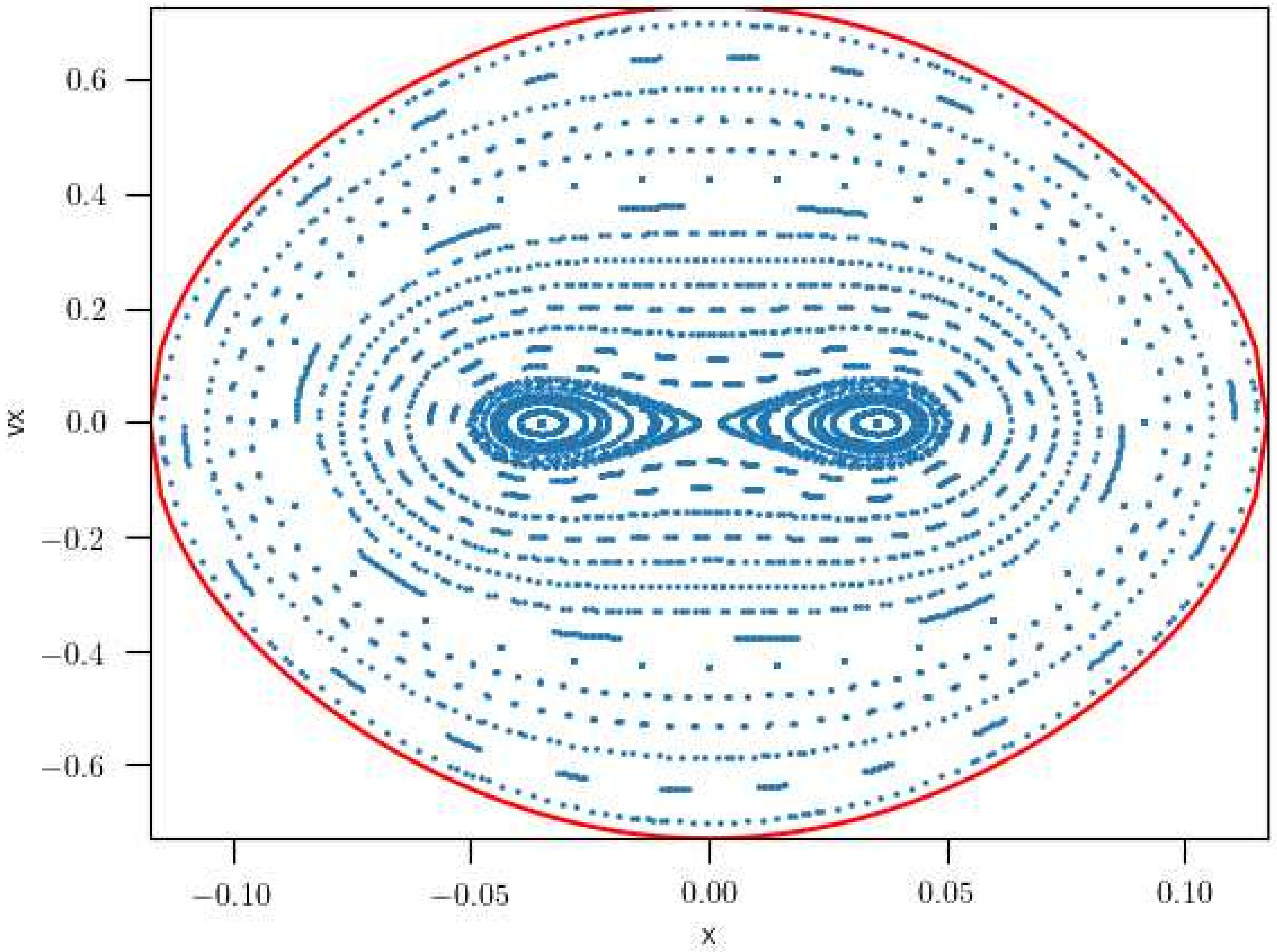

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.80  --norbits 50
```

$E = -1.75$



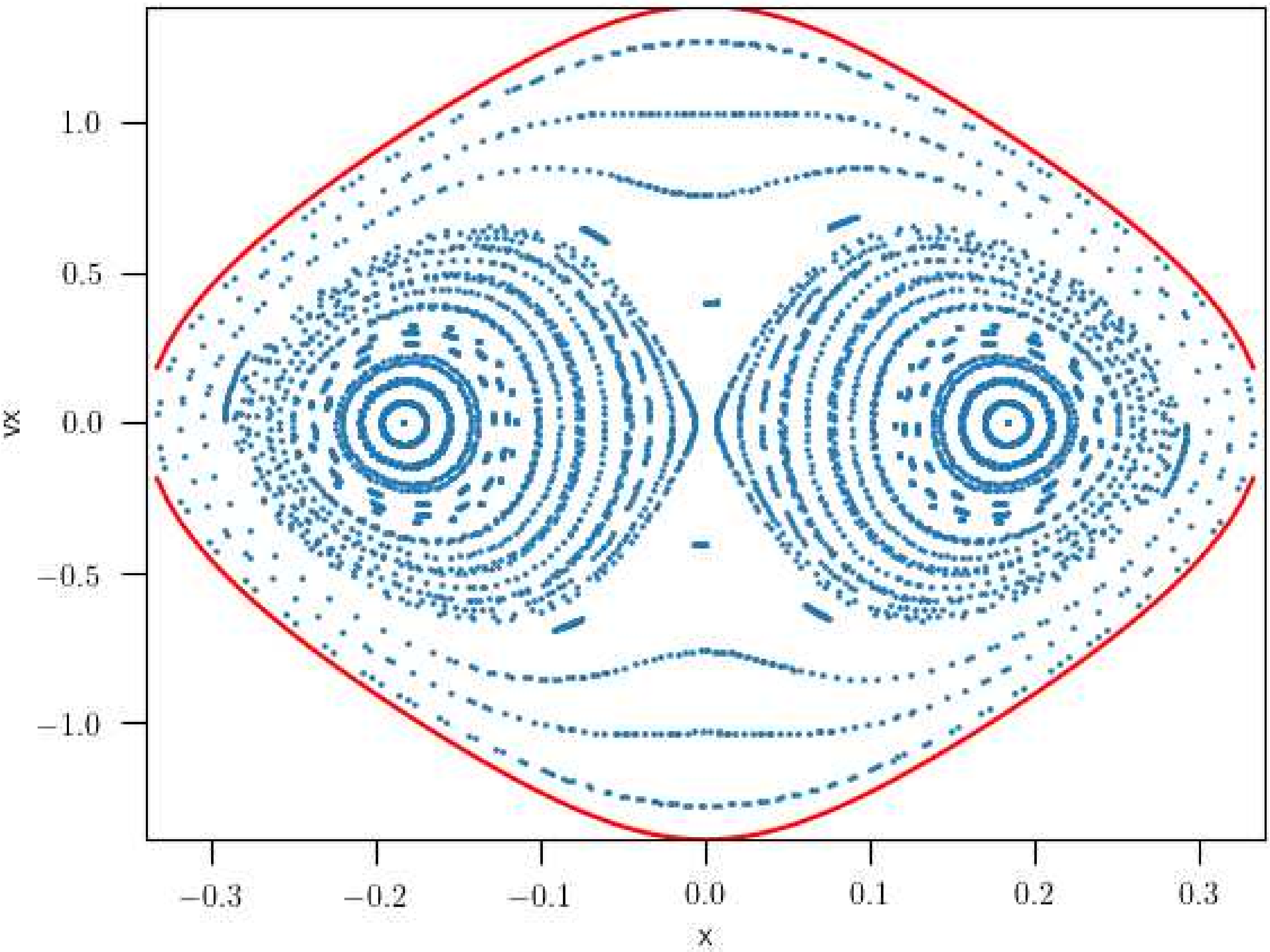

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.75  --norbits 50
```

bifurcation $E = -1.70$



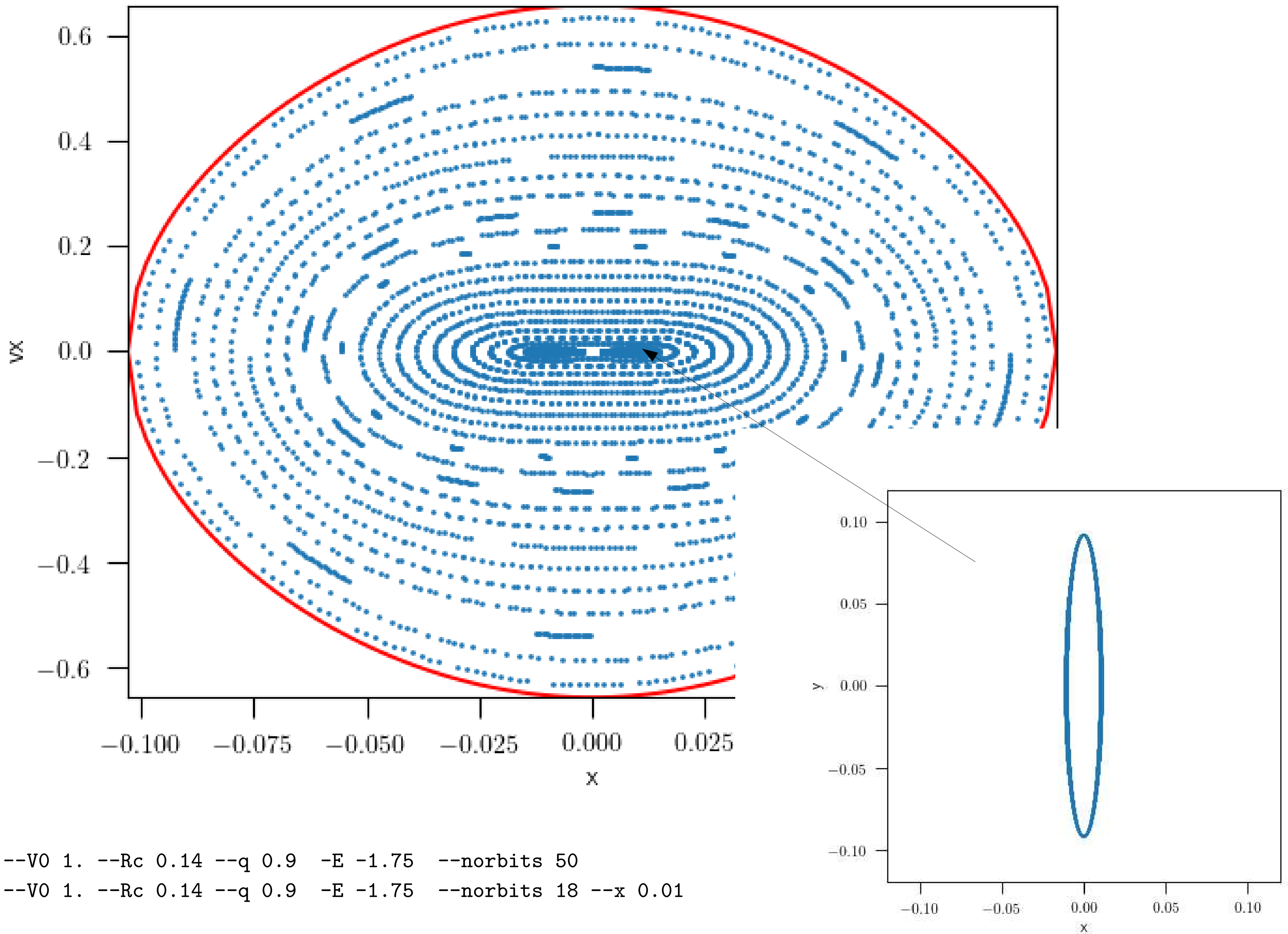

```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.70  --norbits 50
```

$$E = -1.$$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1     --norbits 50
```

$$E = -1.75$$

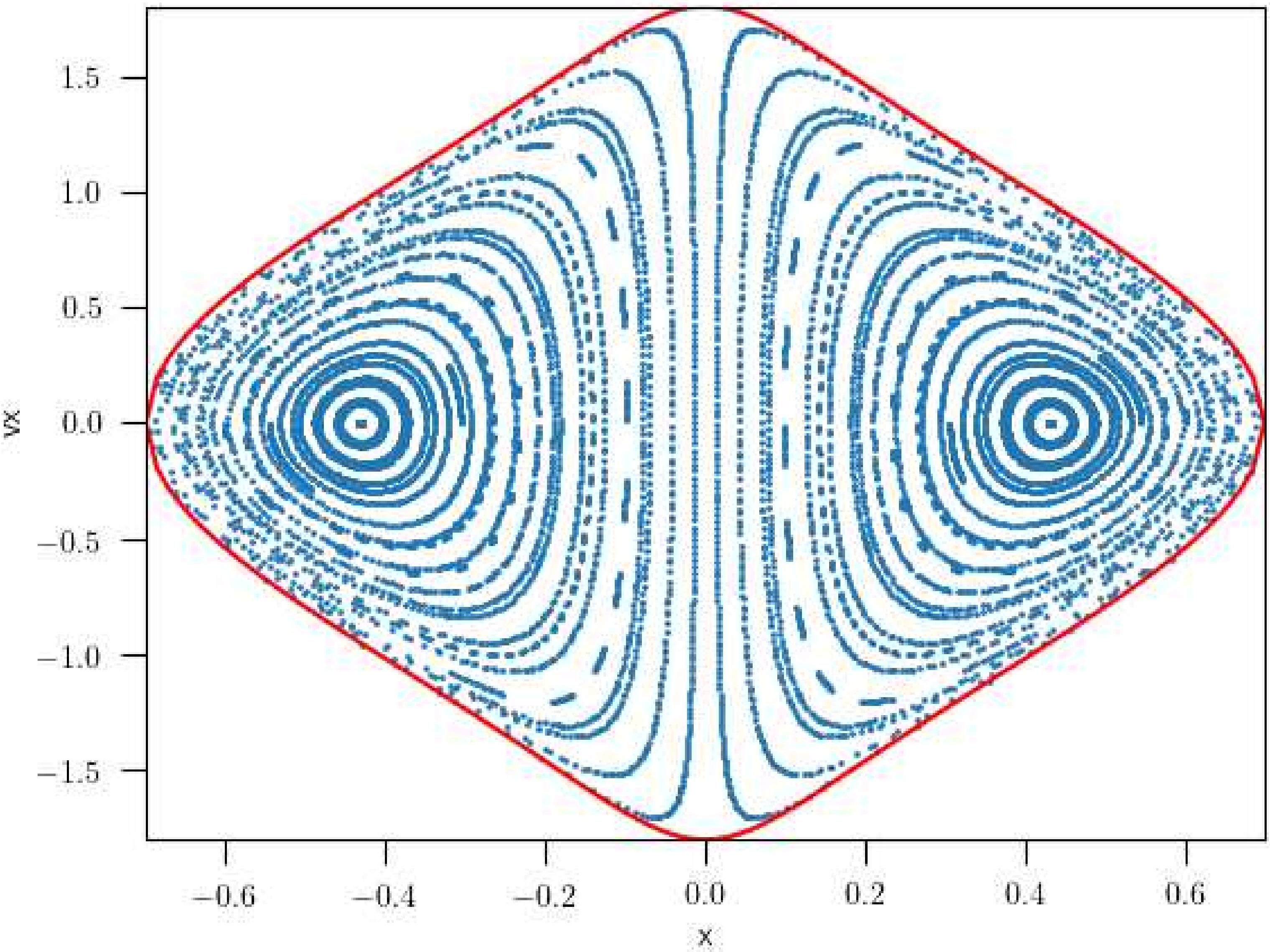



```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.75  --norbits 50
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -1.75  --norbits 18 --x 0.01
```

Evolution with the flattening

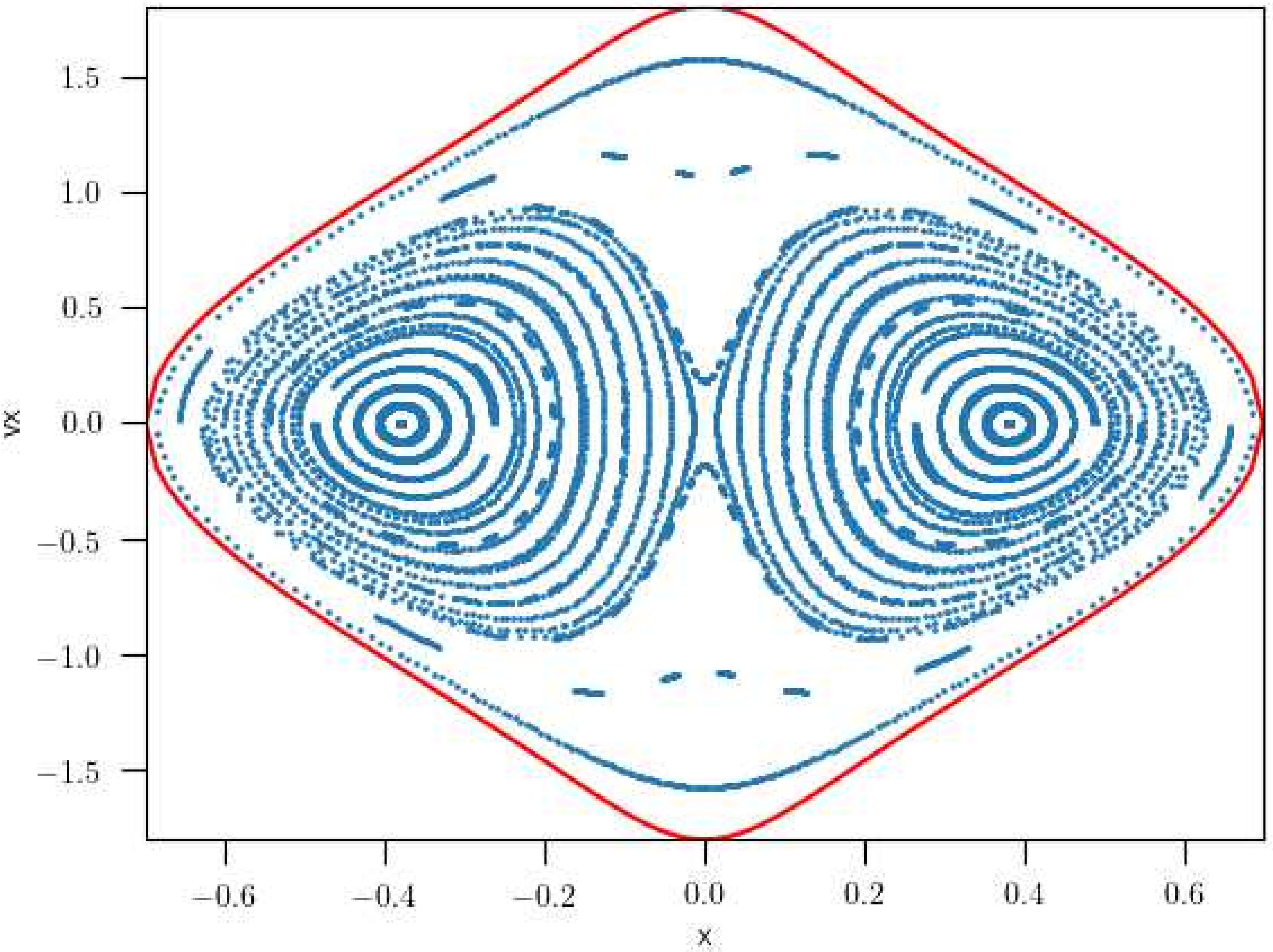
keeping the energy fixed

$q = 1.0$




```
./mapping.py --V0 1. --Rc 0.14 --q 1.0  -E -0.337  --norbits 50  --nlaps 200
```

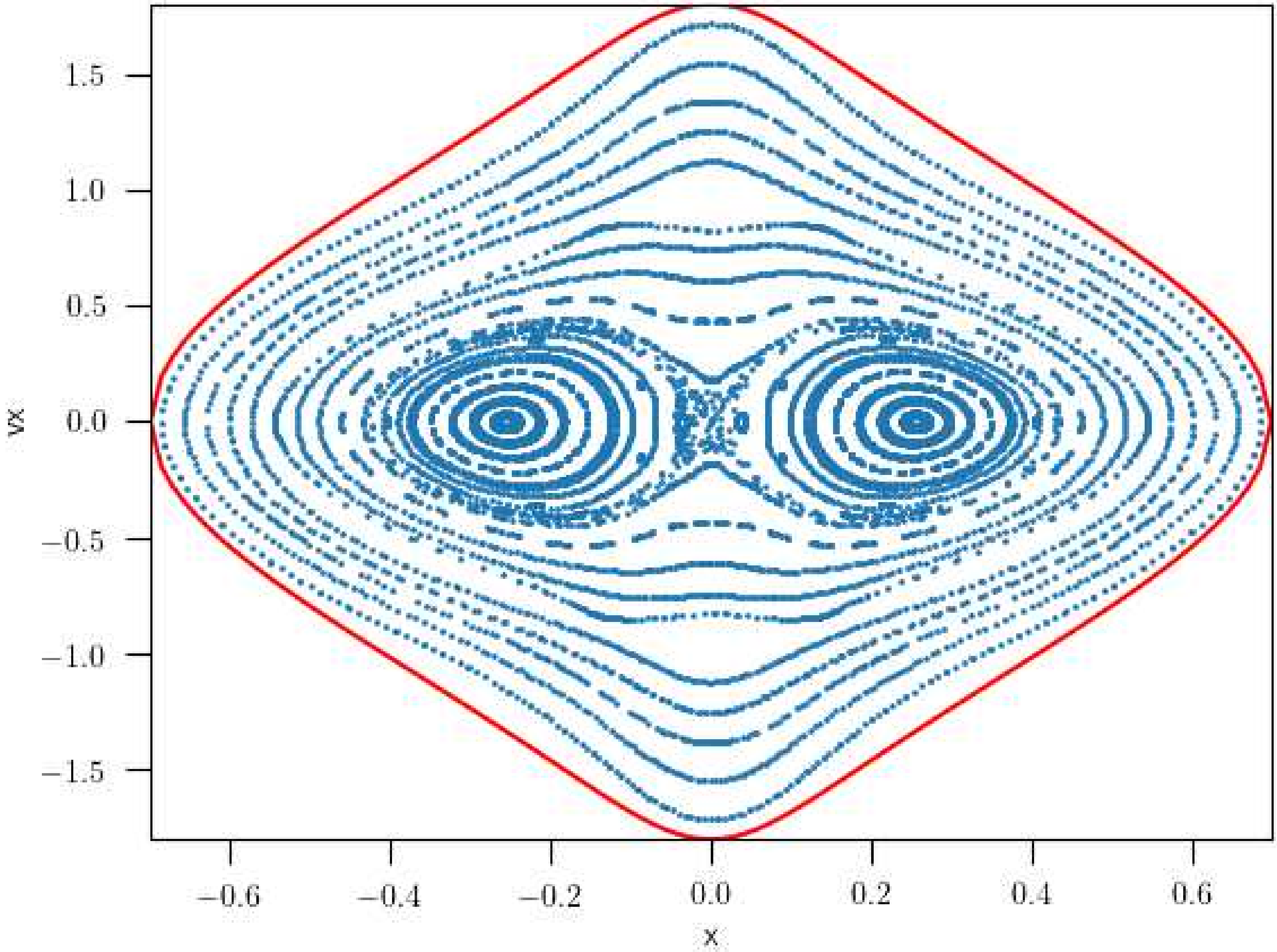
$$q = 0.9$$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.9  -E -0.337  --norbits 50  --nlaps 200
```

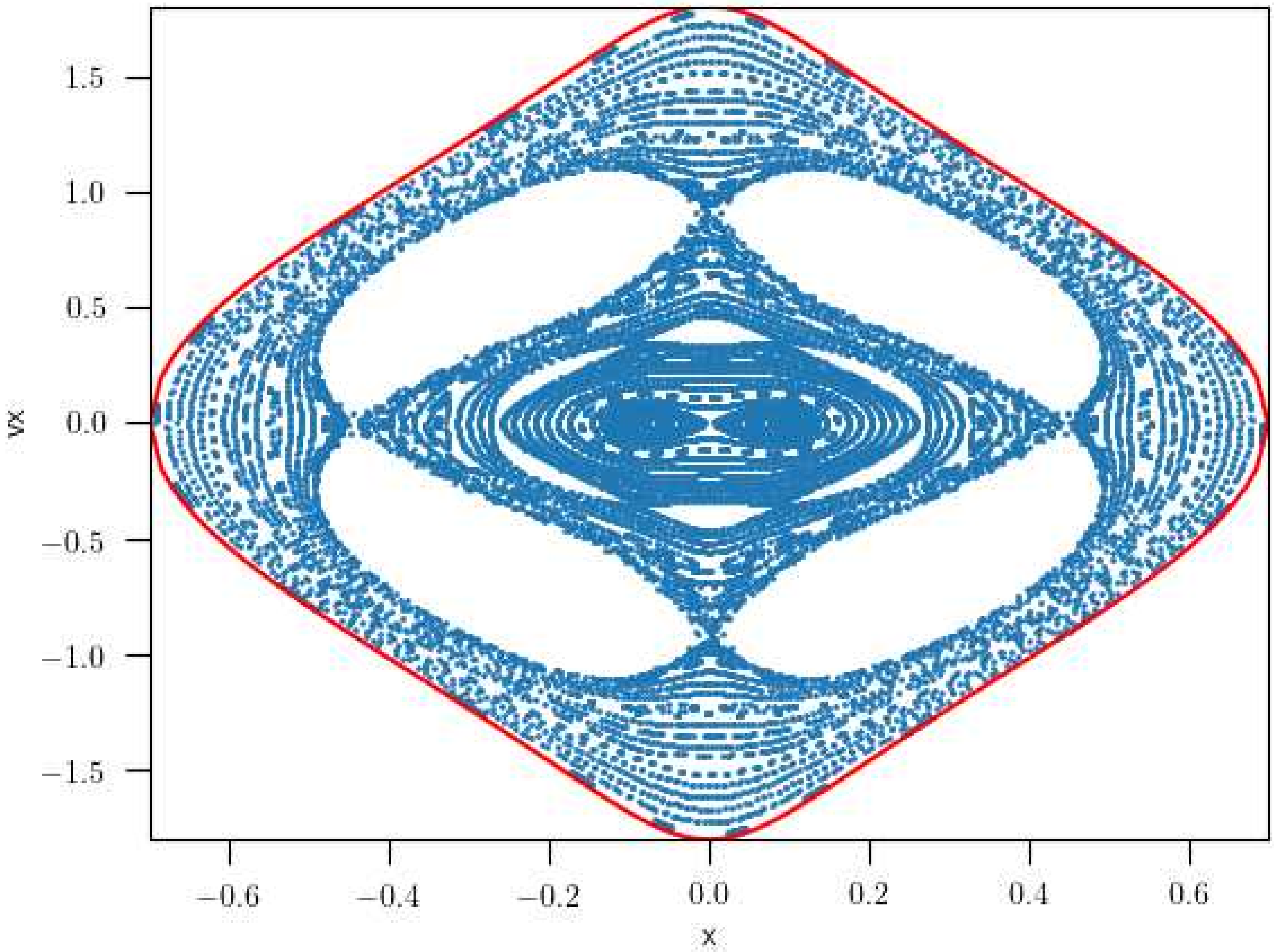
$$q = 0.7$$

Box orbits dominate the phase space !



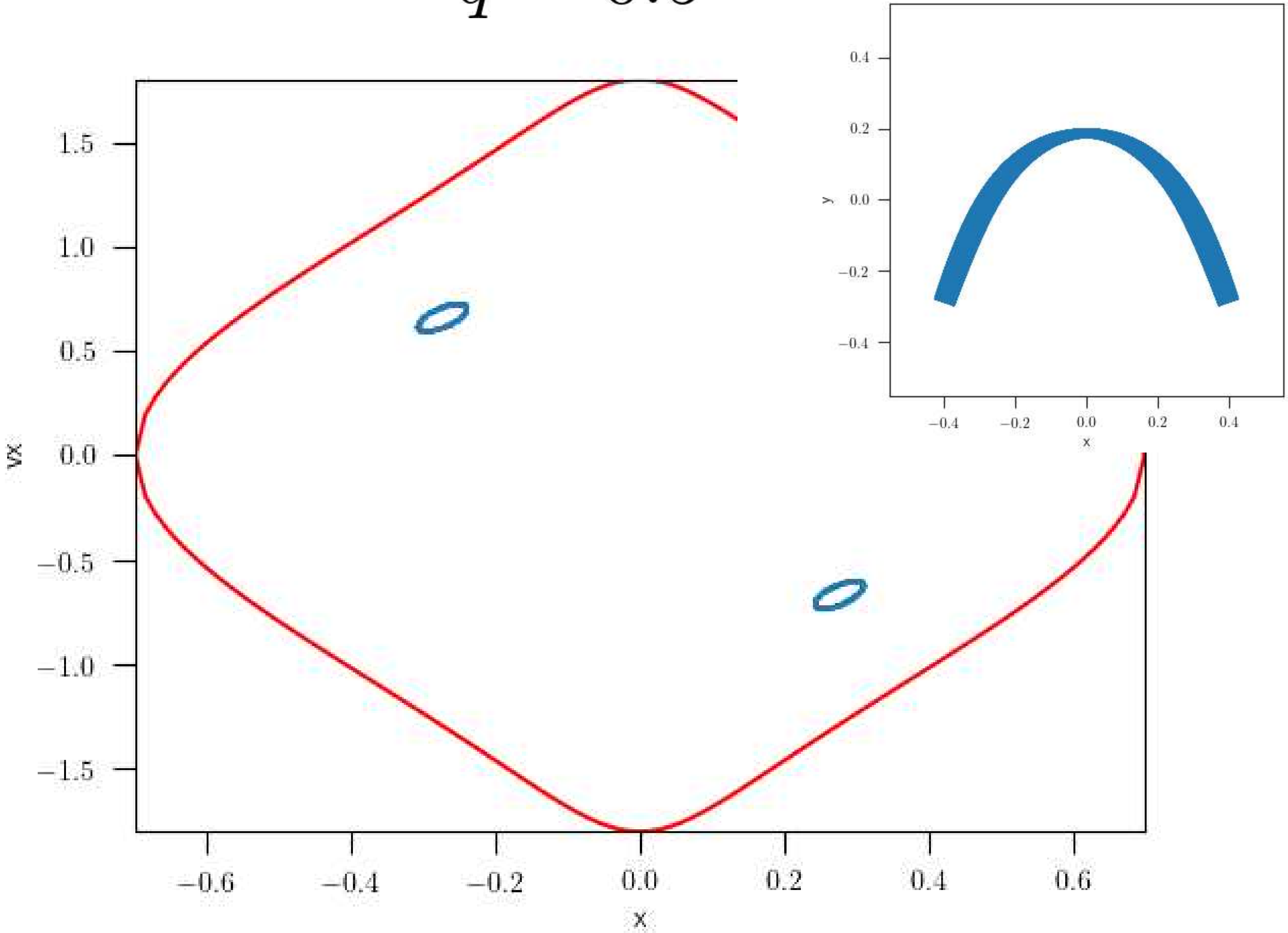

```
./mapping.py --V0 1. --Rc 0.14 --q 0.7  -E -0.337  --norbits 50  --nlaps 200
```

$q = 0.5$



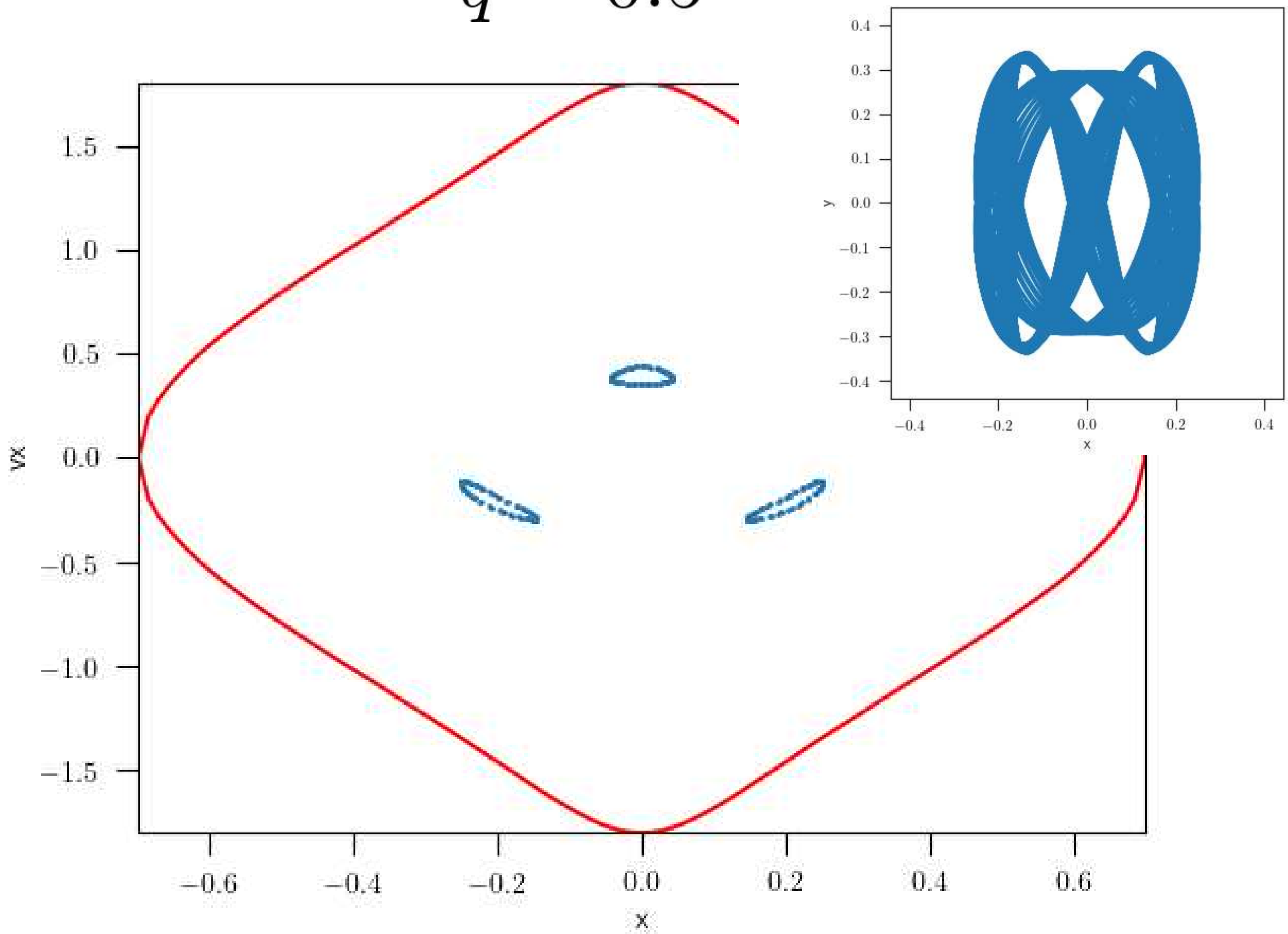

```
./mapping.py --V0 1. --Rc 0.14 --q 0.5  -E -0.337  --norbits 100  --nlaps 200
```

$q = 0.5$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.5  -E -0.337  --norbits 50  --nlaps 200 --x 0.3 --vx -0.6
```

$q = 0.5$




```
./mapping.py --V0 1. --Rc 0.14 --q 0.5  -E -0.337  --norbits 50  --nlaps 200 --x 0.3 --vx -0.6
```

Conclusions

Many 2D bared potential have orbital structures like the logarithmic potential:

- Most orbits respect a 2nd integral (L_z or H_x)
- 2 types of orbits:
  - Loop : - fixed sense of rotation
    - never reach the centre
  - Box : - no fixed sense of rotation
    - many reach the centre

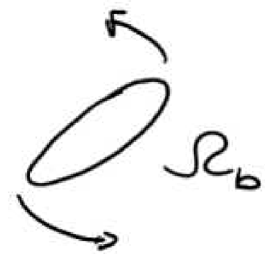
Loop orbits dominate when the axis ratio of of the potential is nearly unity.
Box orbits dominate instead.

Orbits in planar non-axisymmetric rotating potentials

Two dimentional rotating potential

$$\phi(\theta, t) \quad \begin{cases} \theta \to L_z \neq \text{cte} \\ t \to E \neq \text{cte} \end{cases}$$

Assume a static rotation of the bar at constant angular frequency Ω_b



Idea : Describe the motion from the rotating frame where the bar is static

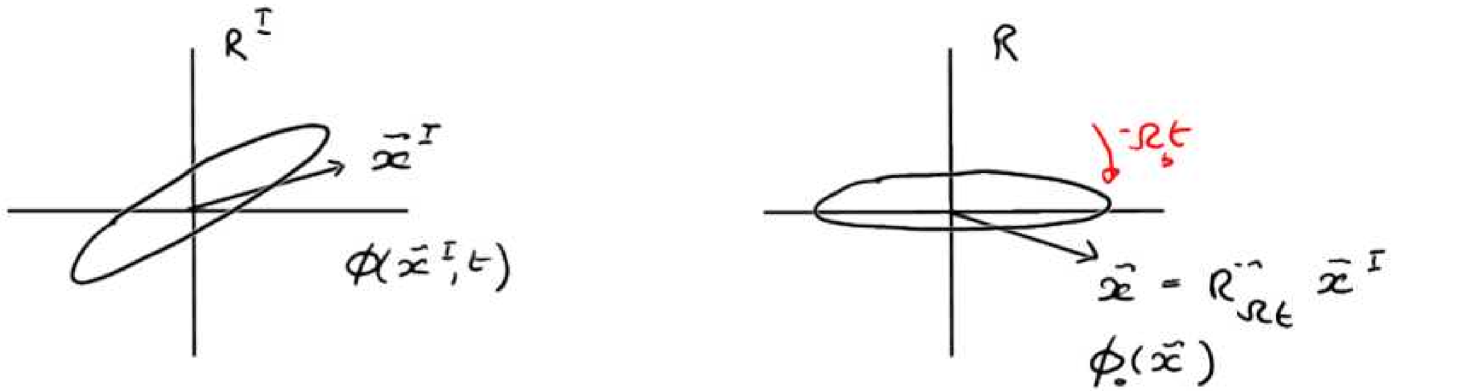
$$(\vec{x}^I, \dot{\vec{x}}^I) \to (\vec{x}, \dot{\vec{x}})$$

inertial frame R^I		rotating frame R

Positions

$$\vec{x} = R^{-1}_{\Omega_b t}\, \vec{x}^{I}$$

$R^{-1}_{\Omega_b t}$: brings the bar to its original position $(t=0)$



Potential

$$\phi(\vec{x}^I, t) \equiv \phi(\vec{x} = R^{-1}_{\Omega t}\, \vec{x}^I, t=0) = \phi_0(\vec{x})$$

$$\phi(\vec{x}^I, t) = \phi_0(\vec{x})$$

Velocities

$$\dot{\vec{x}}^I = \dot{\vec{x}} + \vec{\Omega}_b \times \vec{x}$$

Lagrangian

In the inertial frame R^I

$$\boxed{\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2} \dot{\vec{x}}^{I\,2} - \phi^I(\vec{x}^I, t)}$$

In the rotating frame R

- $\frac{1}{2} \dot{\vec{x}}^{I\,2} \rightarrow \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_b \times \vec{x}\right)^2$
- $\phi^I(\vec{x}^I, t) \rightarrow \phi(R_{\Omega t}^{-1}\, \vec{x}\,,\, t=0) \equiv \phi_0(\vec{x})$

$$\boxed{\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_s \times \vec{x}\right)^2 - \phi_0(\vec{x})}$$

Hamiltonian

$$H_J(\vec{x}, \vec{p}) = \frac{1}{2}\vec{p}^{\,2} - \vec{\Omega}\cdot(\vec{p}\times\vec{x}) + \phi(\vec{x})$$

EXERCICE

H_J has no explicit time dependency

$\Rightarrow$ $H_J = E_J = \text{cte}$

Jacobi integral

Equations of motion from Hamilton's equations

$$H_J = \frac{1}{2}\vec{p}^{\,2} - \vec{\Omega}\cdot(\vec{p}\times\vec{x}) + \phi(\vec{x})$$

$$\dot{\vec{x}} = \frac{\partial H_J}{\partial \vec{p}} = \vec{p} - \vec{\Omega}\times\vec{x}$$

$$\dot{\vec{p}} = -\frac{\partial H_J}{\partial \vec{x}} = -\vec{\nabla}\phi - \vec{\Omega}\times\vec{p}$$

EXERCICE

Effective potential

split the kinetic term in the Lagrangian

$$\mathcal{L}(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}\left(\dot{\vec{x}} + \vec{\Omega}_s \times \vec{x}\right)^2 - \phi_0(\vec{x})$$

$$\equiv \frac{1}{2}\dot{\vec{x}}^2 + \dot{\vec{x}}\left(\vec{\Omega}_s \times \vec{x}\right) \underbrace{- \phi_0(\vec{x}) + \frac{1}{2}\left(\vec{\Omega}_s \times \vec{x}\right)^2}_{\text{depends only on } \vec{x}}$$

$$\phi_{eff}(\vec{x}) := \phi(\vec{x}) - \frac{1}{2}\left(\vec{\Omega} \times \vec{x}\right)^2$$

$$= \phi(\vec{x}) \underbrace{- \frac{1}{2}\vec{\Omega}^2\vec{x}^2 + \frac{1}{2}\left(\vec{\Omega}\cdot\vec{x}\right)^2}$$

$\phi_{centr}(\vec{x})$: repulsive centrifugal potential

Note : $\phi_{centr}(\vec{x}) = -\frac{1}{2}\vec{\Omega}^2\vec{x}^2 + \frac{1}{2}\left(\vec{\Omega}\cdot\vec{x}\right)^2$

Equations of motion from the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} - \frac{\partial \mathcal{L}}{\partial \vec{x}} = 0$$

$$\ddot{\vec{x}} = -\vec{\nabla}\phi_{eff}(\vec{x}) - 2(\Omega \times \dot{\vec{x}})$$

$$\ddot{\vec{x}} = -\vec{\nabla}\phi(\vec{x}) + \Omega^2\vec{x} - \vec{\Omega}(\vec{\Omega}\cdot\vec{x}) - 2(\Omega \times \dot{\vec{x}})$$

centrifugal force $\quad$ Coriolis force

$$= \Omega^2 \vec{x} \quad \text{if } \vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$$ and z=0

Stationary points

$\dot{\vec{x}} = \ddot{\vec{x}} = 0$ (in the rotating frame)

1) The point co-rotate with the bar ($V_\perp = \Omega R$)

2) with $\ddot{\vec{x}} = -\vec{\nabla}\phi_{eff} - 2\vec{\Omega}\times\underbrace{\dot{\vec{x}}}_{=0} = 0$

$$\vec{\nabla}\phi_{eff} = 0$$

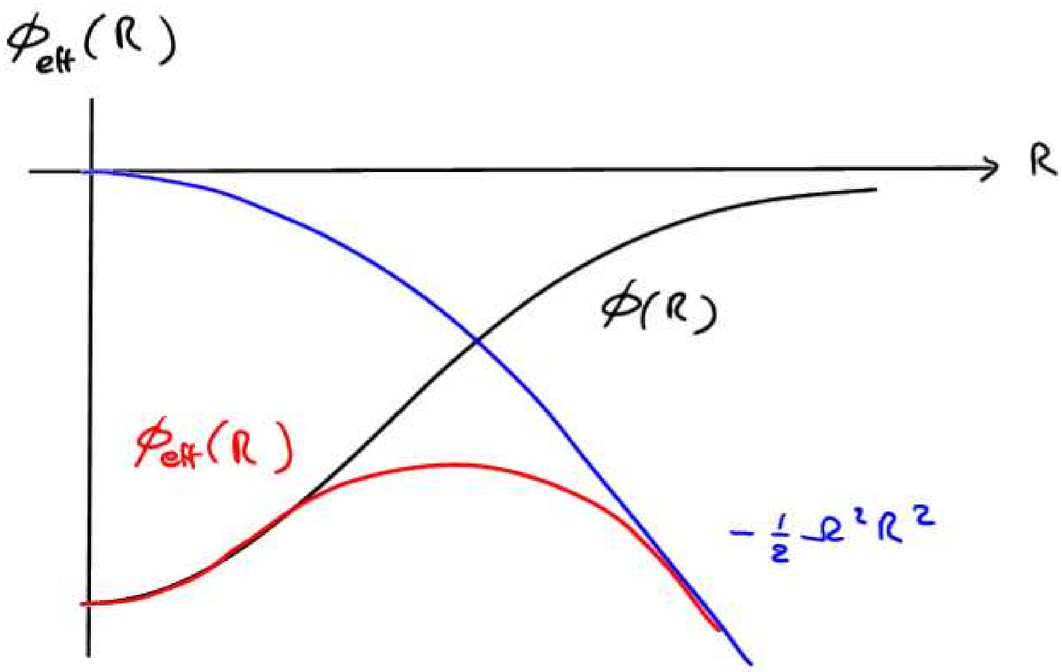
"gravity" counter balance the centrifugal force

Positions of the stationary points

1) assume $y = 0$ $\quad \vec{\Omega}_b = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$

2) assume an axi-symetric system

$$\phi_{eff}(\vec{x}) \rightarrow \quad \phi_{eff}(R) = \phi(R) - \frac{1}{2}\Omega^2 R^2$$

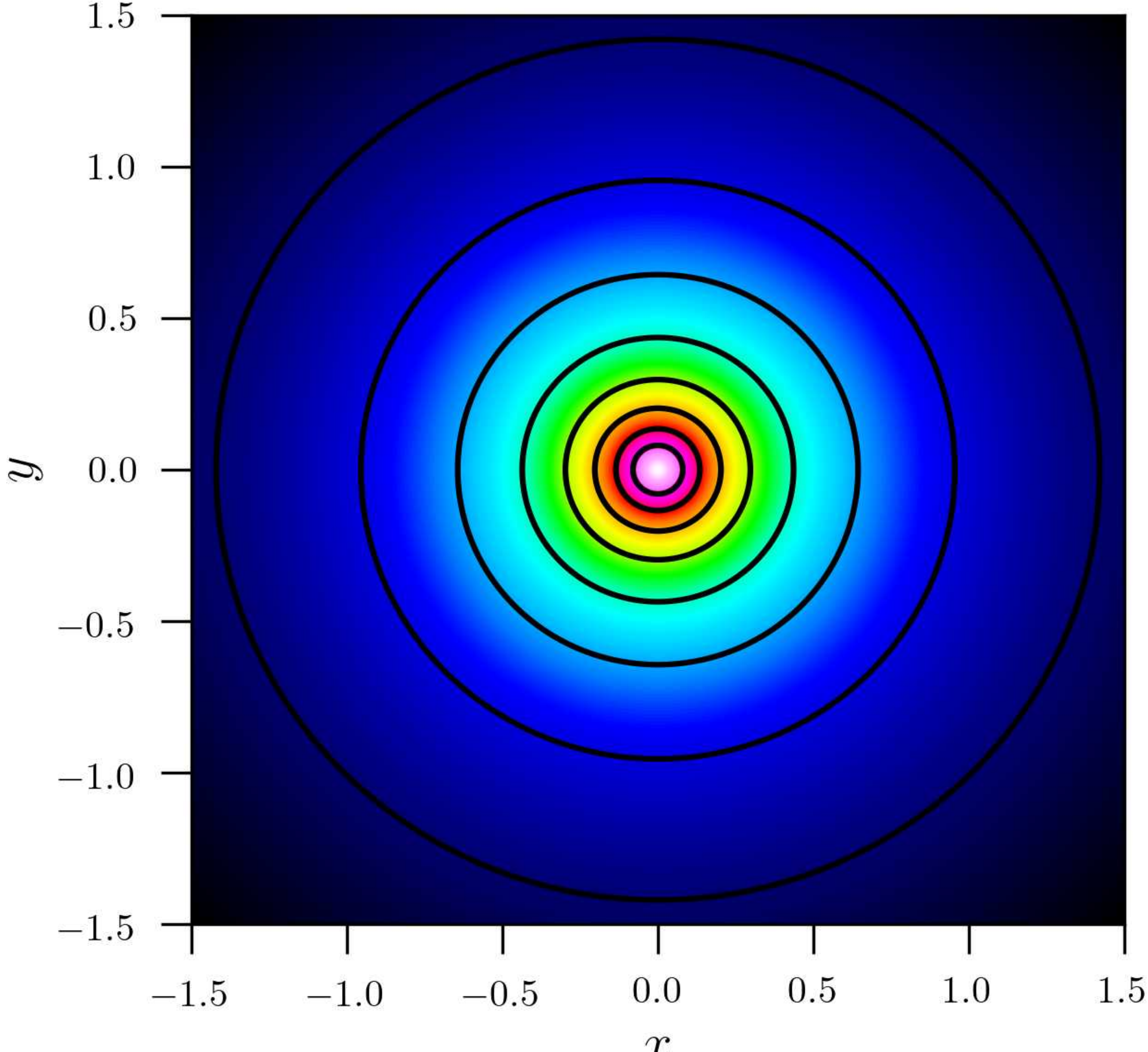


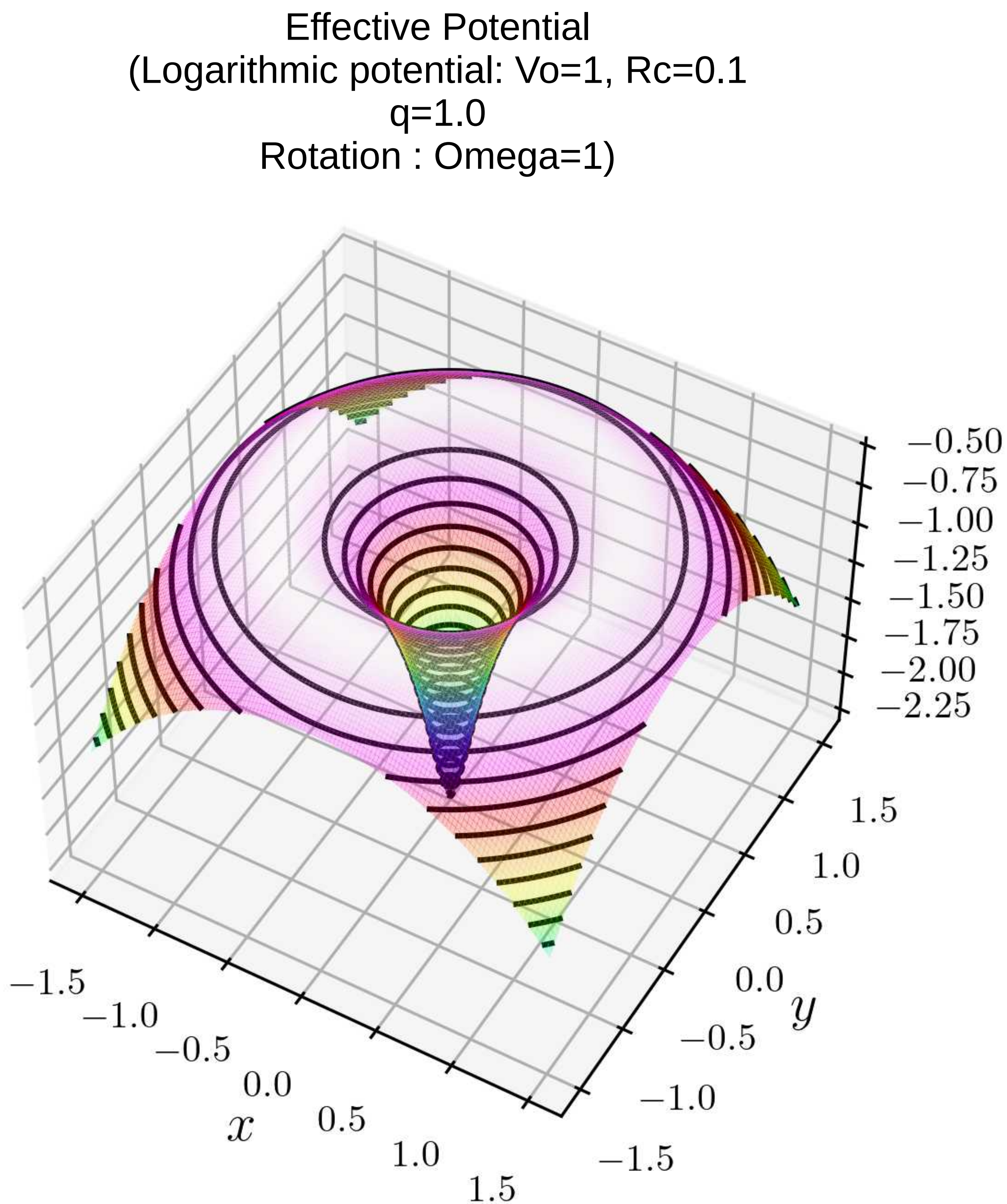
$$\vec{\nabla}\phi_{eff}(\vec{x}) = 0 \quad \equiv \quad \frac{\partial}{\partial R}\phi_{eff}(R) = 0$$

$$\frac{\partial}{\partial R}\phi(R) = \Omega^2 R$$

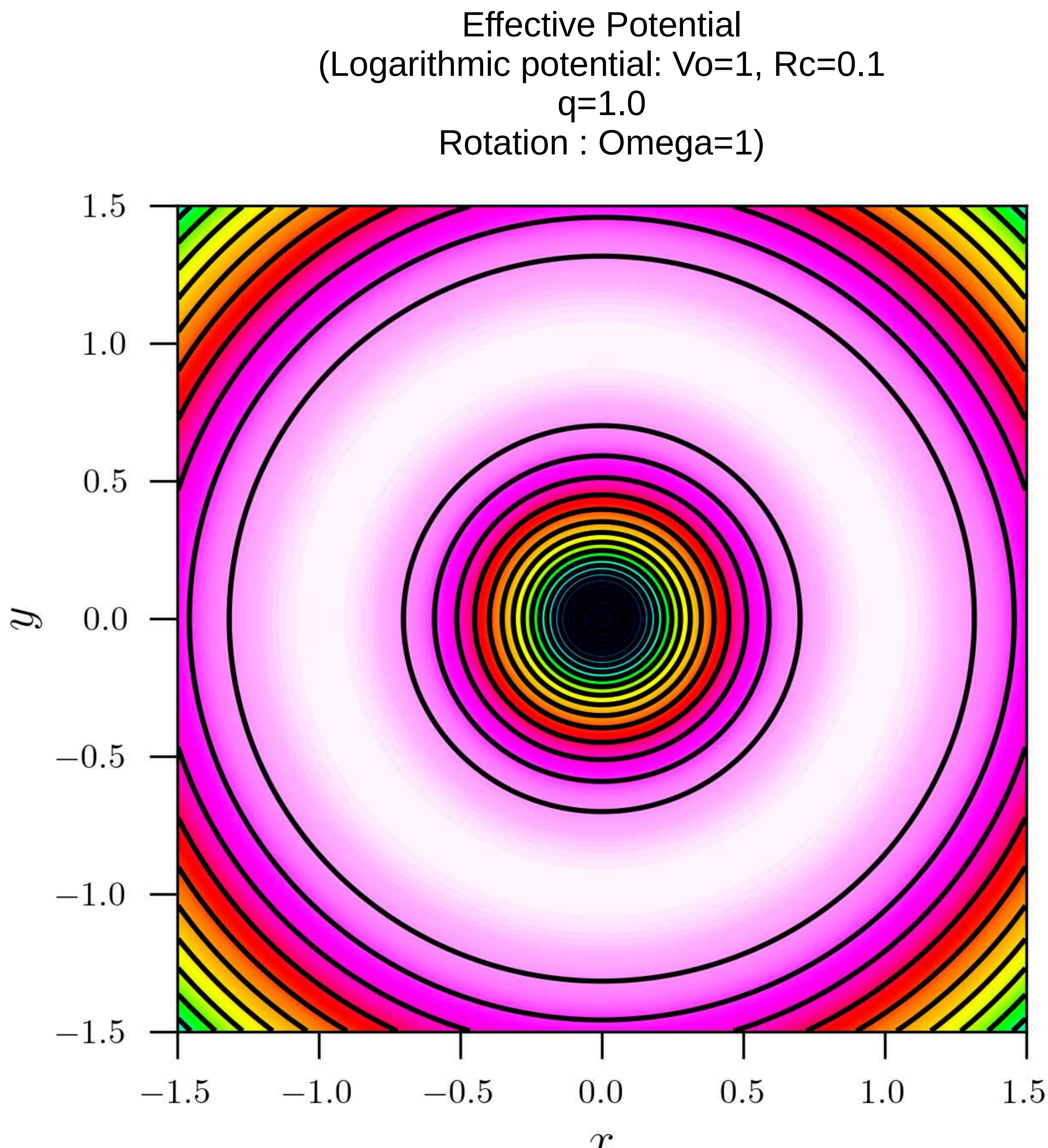
$\Rightarrow$ selection of a circular orbit in the inertial frame

Bar potential (density)
(Logarithmic potential: Vo=1, Rc=0.1
q=1.0)

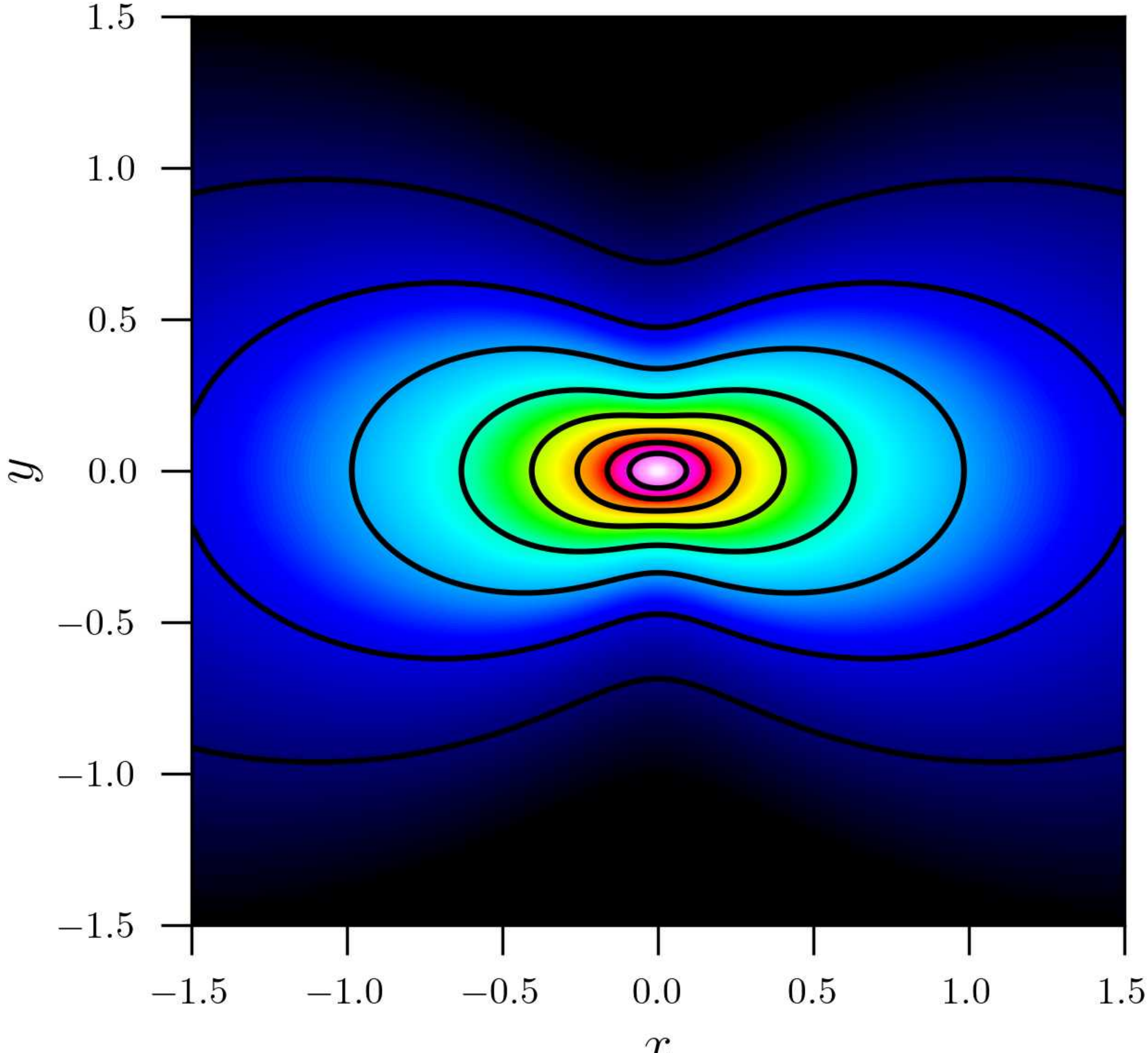




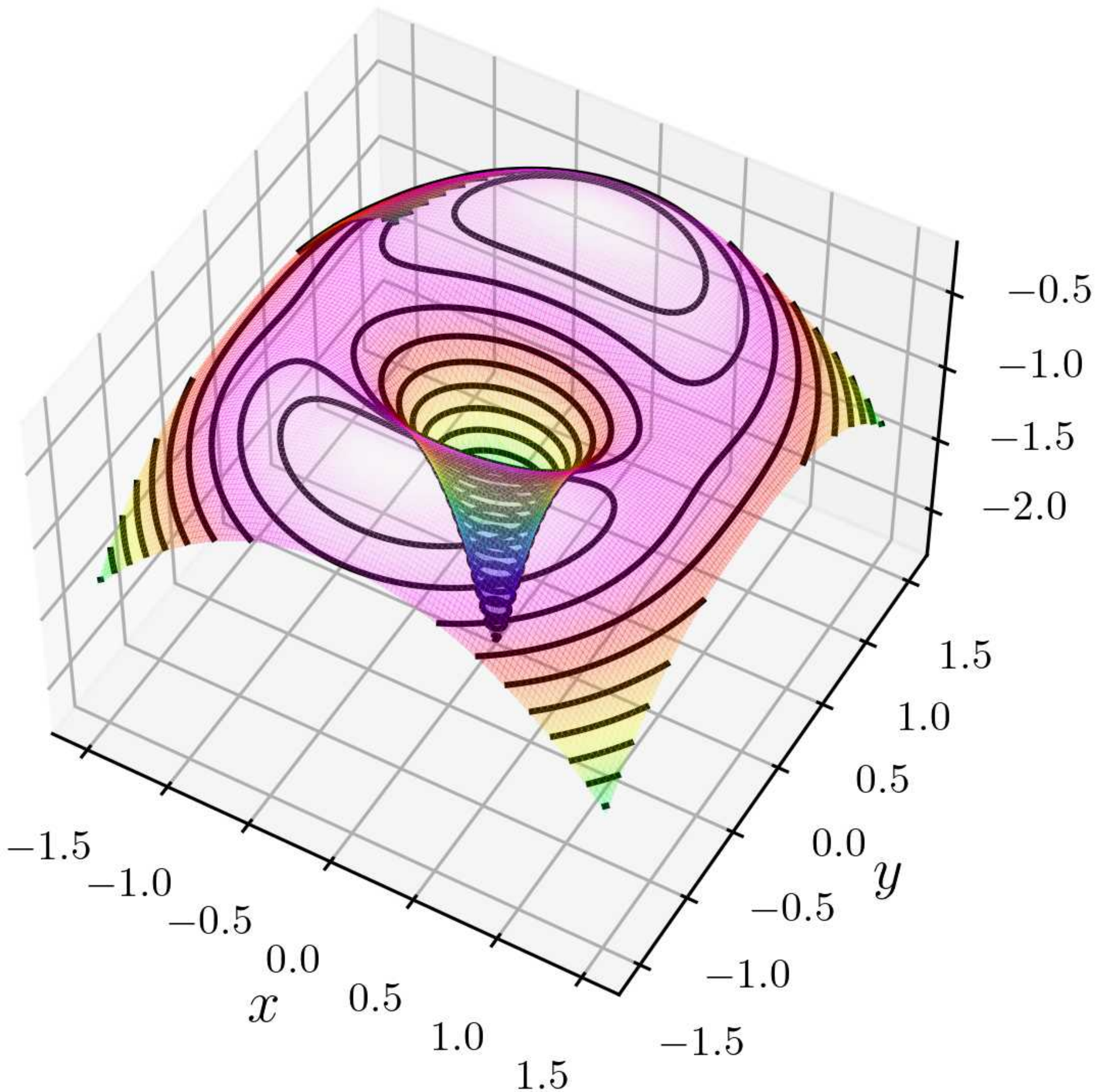
Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=1.0
Rotation : Omega=1)
−0.50
−0.75
−1.00
−1.25
−1.50
−1.75
−2.00
−2.25
x
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
y
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5


Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=1.0
Rotation : Omega=1)
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
x

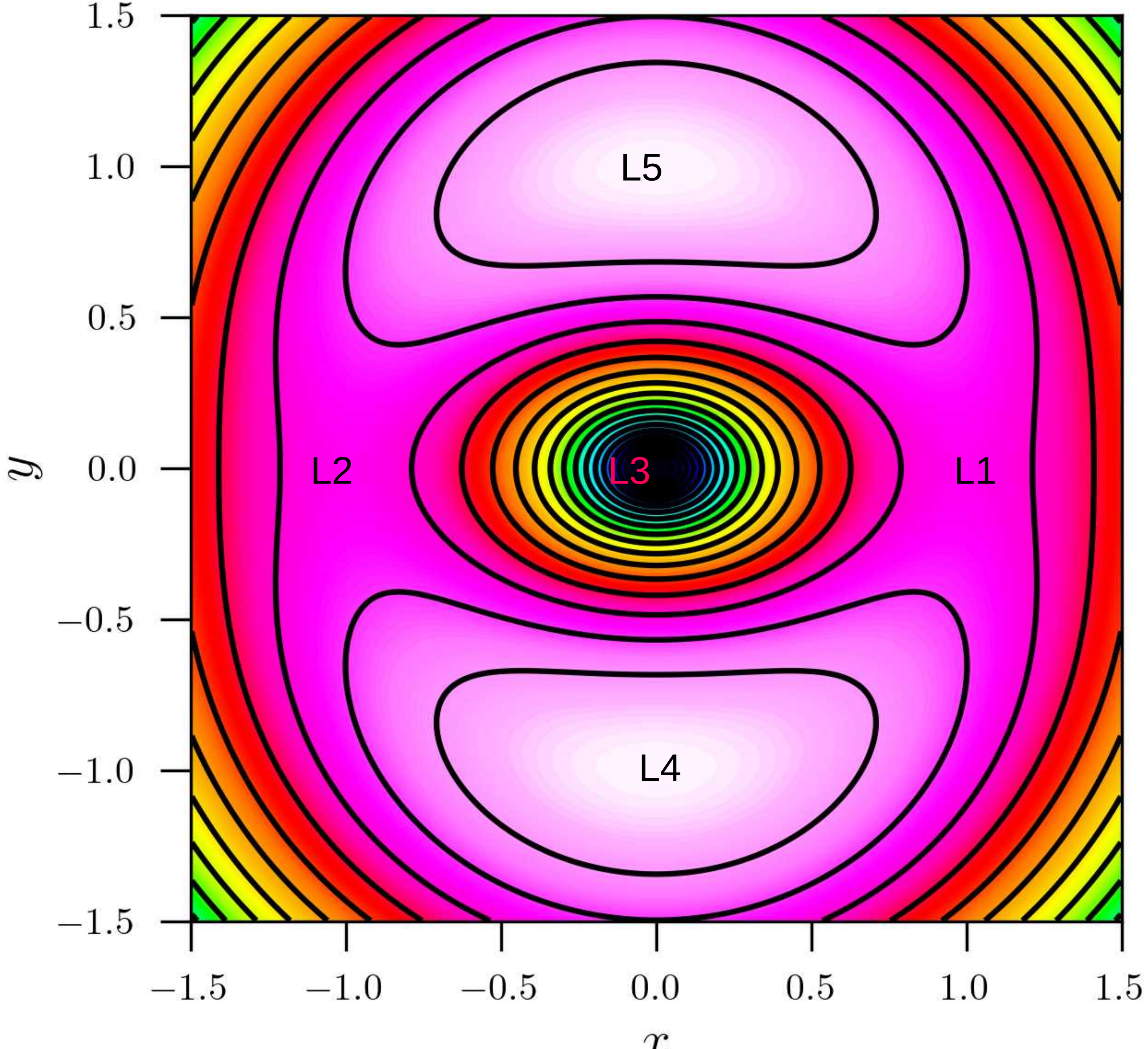
Bar potential
(Logarithmic potential: Vo=1, Rc=0.1
q=0.75)



Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=0.75
Rotation : Omega=1)



Effective Potential
(Logarithmic potential: Vo=1, Rc=0.1
q=0.75
Rotation : Omega=1)



Orbits around Lagrange points

Stability of orbits around Lagrange points

Expand the effective potential in Taylor serie around the Lagrange points (x_L, y_L)

$$\phi_{eff}(x,y) \cong \phi_{eff}(x_L,y_L) + \underbrace{\frac{\partial \phi_{eff}}{\partial x}}_{=0}(x-x_L) + \underbrace{\frac{\partial \phi_{eff}}{\partial y}}_{=0}(y-y_L)$$

$$+ \frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial x^2}(x-x_L)^2 + \frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial y^2}(y-y_L)^2 + \underbrace{\frac{1}{2}\frac{\partial^2 \phi_{eff}}{\partial x \partial y}}_{=0}(x-x_L)(y-y_L)$$

by symmetry of the bar, if it is aligned with $\bar{x}$

Now we define

$$\xi := x - x_L \qquad \phi_{xx} := \frac{\partial^2 \phi_{eff}}{\partial x^2}$$

$$\eta := y - y_L \qquad \phi_{yy} := \frac{\partial^2 \phi_{eff}}{\partial y^2}$$

$$\phi_{eff}(\xi,\eta) = \phi_{eff}(0,0) + \frac{1}{2}\phi_{xx}\xi^2 + \frac{1}{2}\phi_{yy}\eta^2$$

Equations of motions

$$\ddot{\vec{x}} = -\vec{\nabla}\phi_{\text{eff}} - 2\left(\vec{\Omega} \times \dot{\vec{x}}\right)$$

in the plane $z = 0$ assuming $\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$

$$\begin{cases} \ddot{x} = -\dfrac{\partial \phi_{\text{eff}}}{\partial x} + 2\Omega\dot{y} \\[2ex] \ddot{y} = -\dfrac{\partial \phi_{\text{eff}}}{\partial y} - 2\Omega\dot{x} \end{cases}$$

$$\begin{cases} \ddot{\xi} = +2\Omega\dot{\eta} - \phi_{xx}\xi \\[2ex] \ddot{\eta} = -2\Omega\dot{\xi} - \phi_{yy}\eta \end{cases}$$

We assume solutions of the form

$$\begin{cases} \xi(t) = X e^{\lambda t} \\ \eta(t) = Y e^{\lambda t} \end{cases} \qquad X, Y, \lambda \in \mathbb{C}$$

The EoM become

$$\begin{cases} (\lambda^2 + \phi_{xx}) X - (2\lambda\Omega) Y = 0 \\ (2\lambda\Omega) X + (\lambda^2 + \phi_{yy}) Y = 0 \end{cases}$$

$$\underbrace{\begin{pmatrix} \lambda^2 + \phi_{xx} & -2\lambda\Omega \\ 2\lambda\Omega & \lambda^2 + \phi_{yy} \end{pmatrix}}_{M} \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

simple linear equation

Non trivial solutions (i.e $X \neq 0$, $Y \neq 0$) only if $\mathrm{Det}(M) = 0$

$$\mathrm{Det}\, M = \boxed{\lambda^4 + \lambda^2(\phi_{xx} + \phi_{yy} + 4\Omega^2) + \phi_{xx}\phi_{yy} = 0}$$

"characteristic equation"

Solutions

(4 roots, two are coupled)

- if λ is a solution $\Rightarrow$ $-\lambda$ is a solution
- if λ is real

$$\begin{cases} \xi(t) = X e^{\lambda t} \rightarrow \text{exponential growth} \\ \eta(t) = Y e^{\lambda t} \rightarrow \text{exponential growth} \end{cases}$$

$\rightarrow$ the star leaves the Lagrange point

UNSTABLE

- if all λ are purely complex

$\lambda_1 = \alpha i \quad \lambda_2 = -\alpha i$

$\lambda_3 = \beta i \quad \lambda_4 = -\beta i \qquad \alpha, \beta \in \mathbb{R}$

$$\xi(t) = \mathrm{Re}\left(X_1' e^{i\alpha t} + X_2' e^{-i\alpha t} + X_3' e^{i\beta t} + X_4' e^{-i\beta t} \right)$$

$$= X_1' \cos(\alpha t) + X_2' \cos(-\alpha t) + X_3' \cos(\beta t) + X_4' \cos(-\beta t)$$

$$= X_1 \cos(\alpha t) + X_2 \cos(\beta t)$$

idem for $\eta(t)$, so we get

$$\begin{cases} \xi(t) = X_1 \cos(\alpha t) + X_2 \cos(\beta t) \\ \eta(t) = Y_1 \cos(\alpha t) + Y_2 \cos(\beta t) \end{cases}$$

STABLE

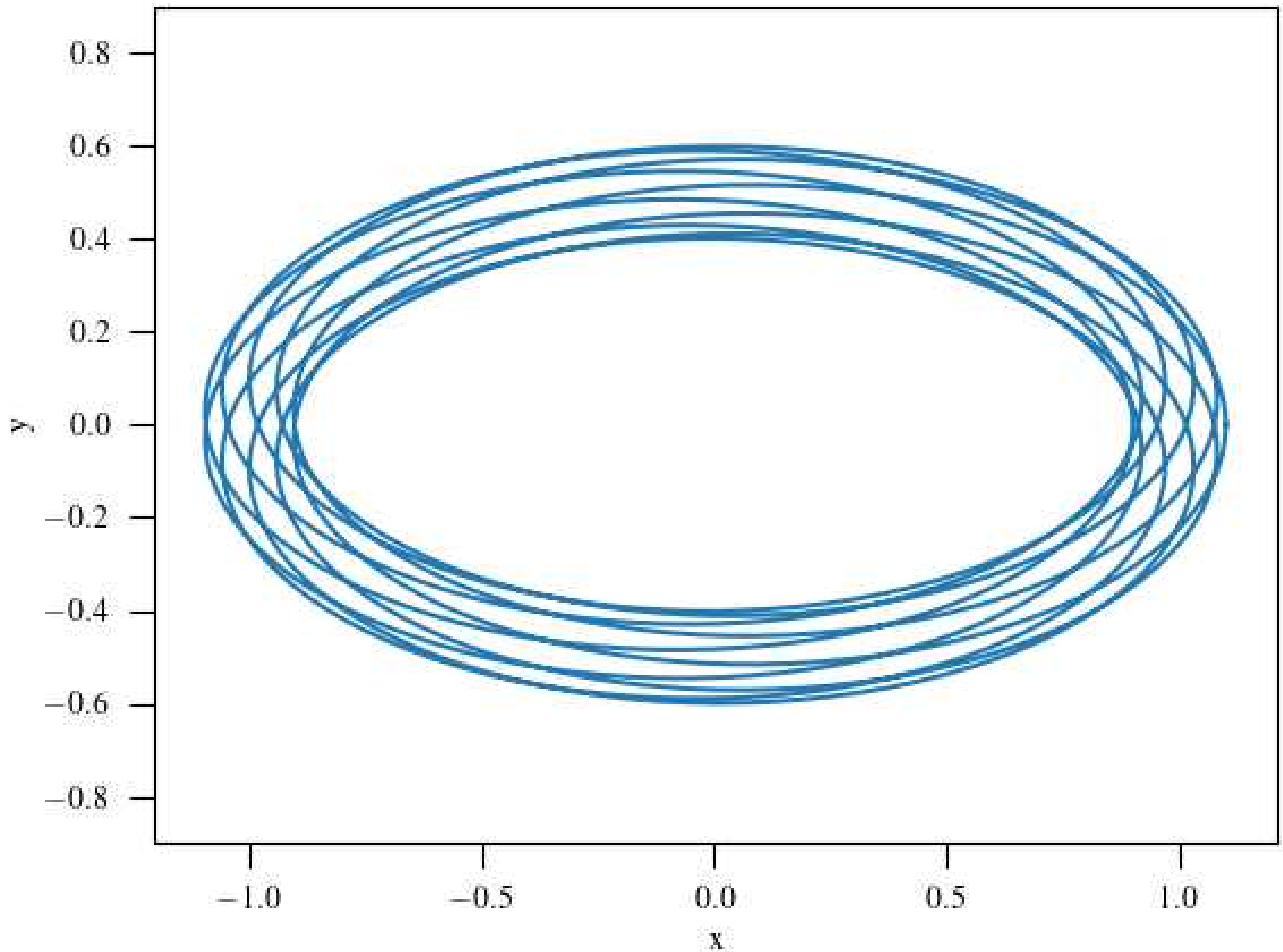
with
$$\begin{cases} Y_1 = \dfrac{\phi_{xx} - \alpha^2}{2\Omega\alpha} X_1 = \dfrac{2\Omega\alpha}{\phi_{yy} - \alpha^2} X_1 \\ Y_2 = \dfrac{\phi_{xx} - \beta^2}{2\Omega\beta} X_1 = \dfrac{2\Omega\beta}{\phi_{yy} - \beta^2} X_2 \end{cases}$$

It is possible to demonstrate that :

- At L_3 i.e $\min(\phi_{eff})$ — **always stable**
- At L_2, L_3 i.e the saddles points — **always unstable**
- At L_4, L_5 i.e $\max(\phi_{eff})$ — **stable or unstable**

depends on the detail of the potential

Note : The stability comes from the Coriolis force (see Padmanabhan)



lways stable

lways unstable

able or unstable

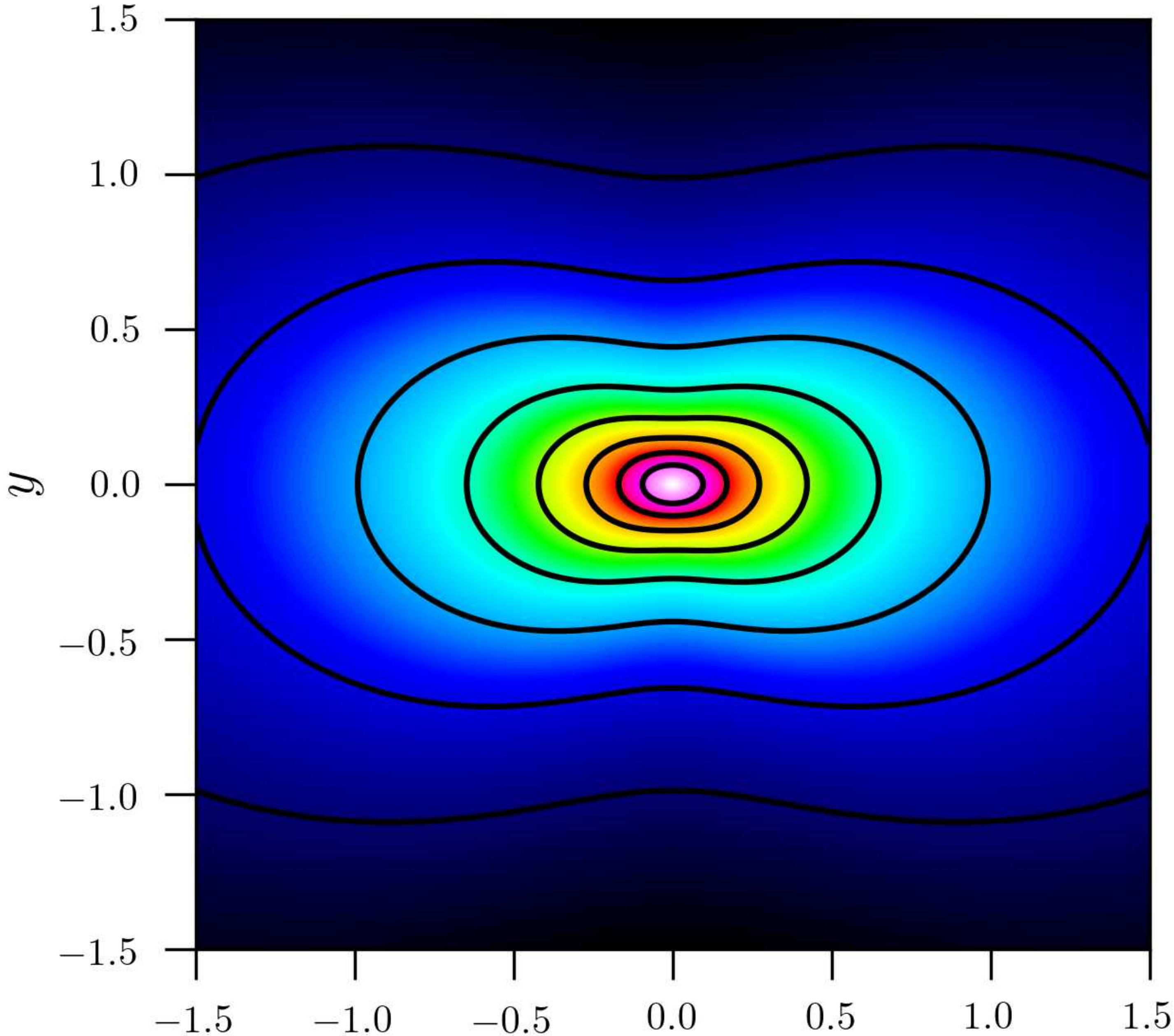
. the detail of
the potential

<u>Note</u>: The stability comes from
the Coriolis force (see Padmanabhan)

Orbits not confined to Lagrange points

Bar model : Logarithmic potential:
Vo=01 Rc=0.1 q=0.8)

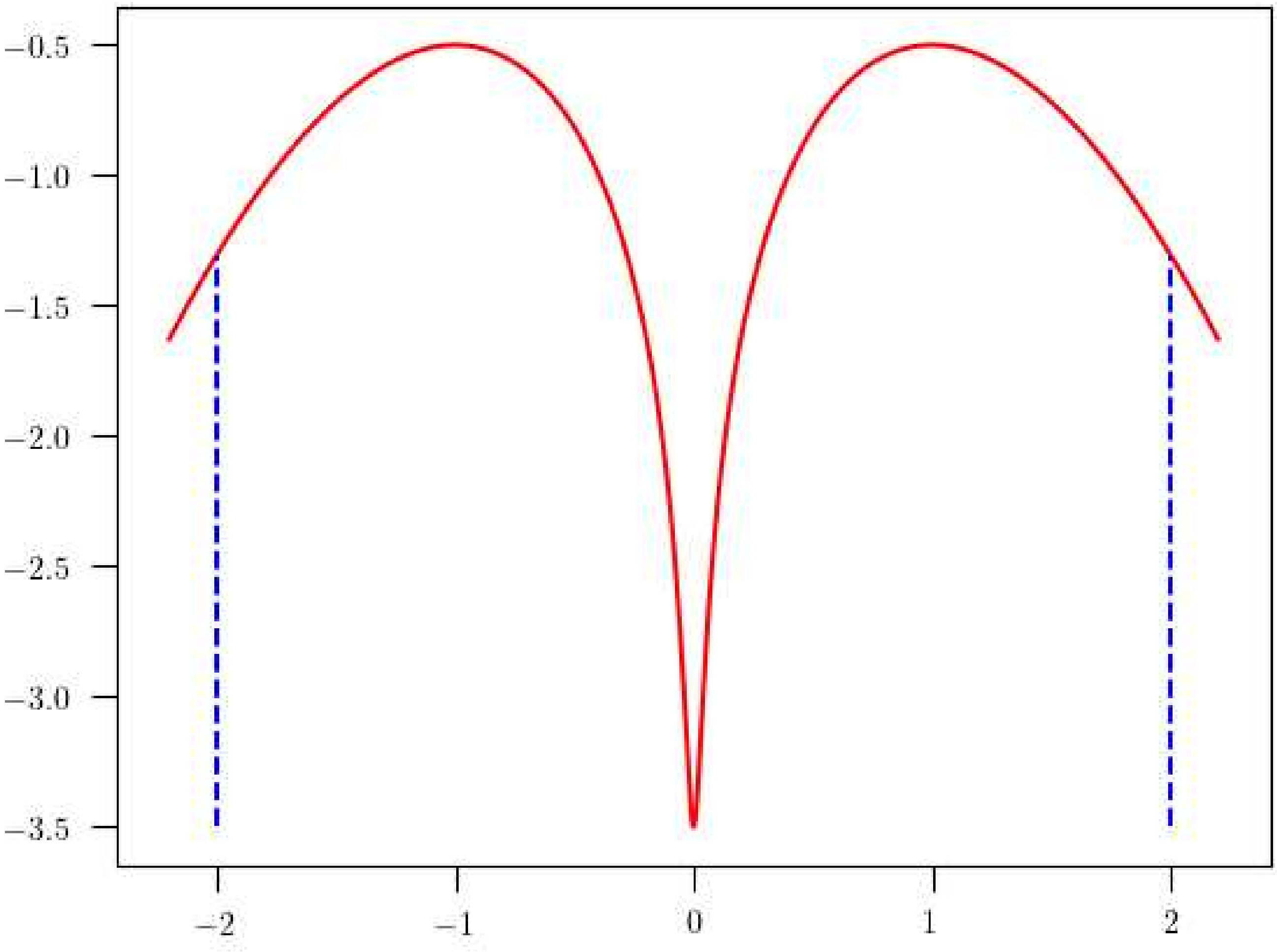
$$\Phi_{\log}(x, y) = \frac{1}{2} V_0^2 \ln \left(R_c^2 + x^2 + \left(\frac{y}{q} \right)^2 \right) \qquad \Omega_{\rm p} \neq 0$$



Low energy orbits

$$R \ll R_{\mathrm{corot}}$$

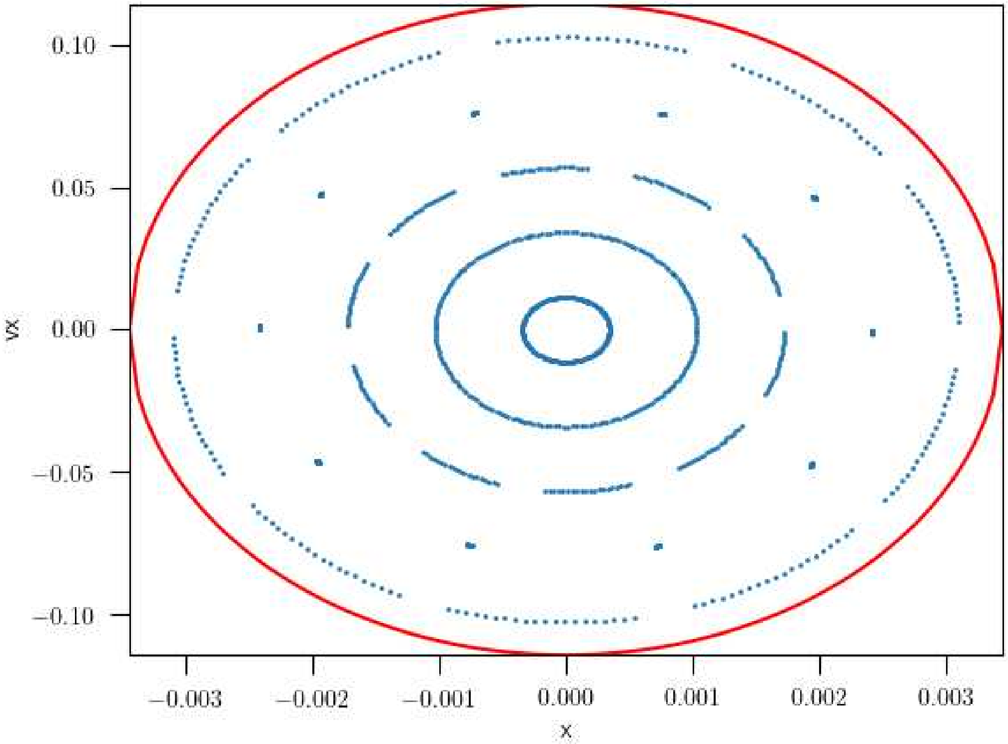
Potential and energy




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

Orbits around L_3

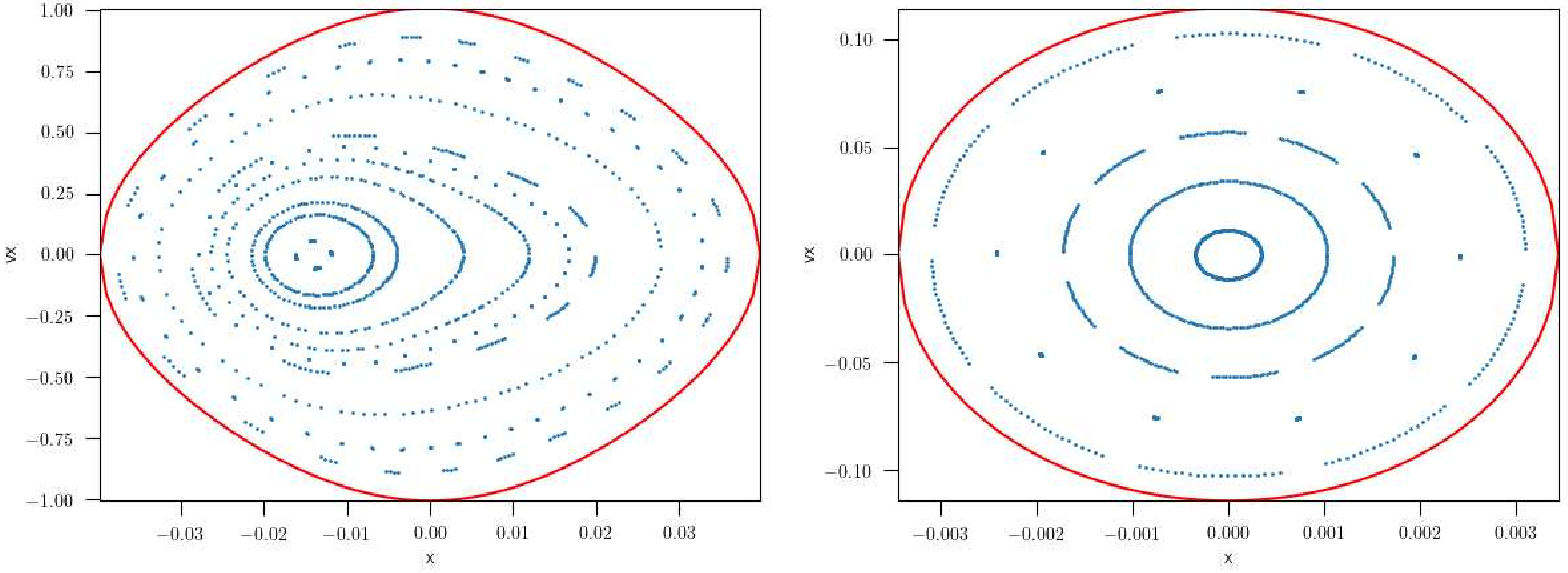
$$\Omega = 0$$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Orbits around L_3

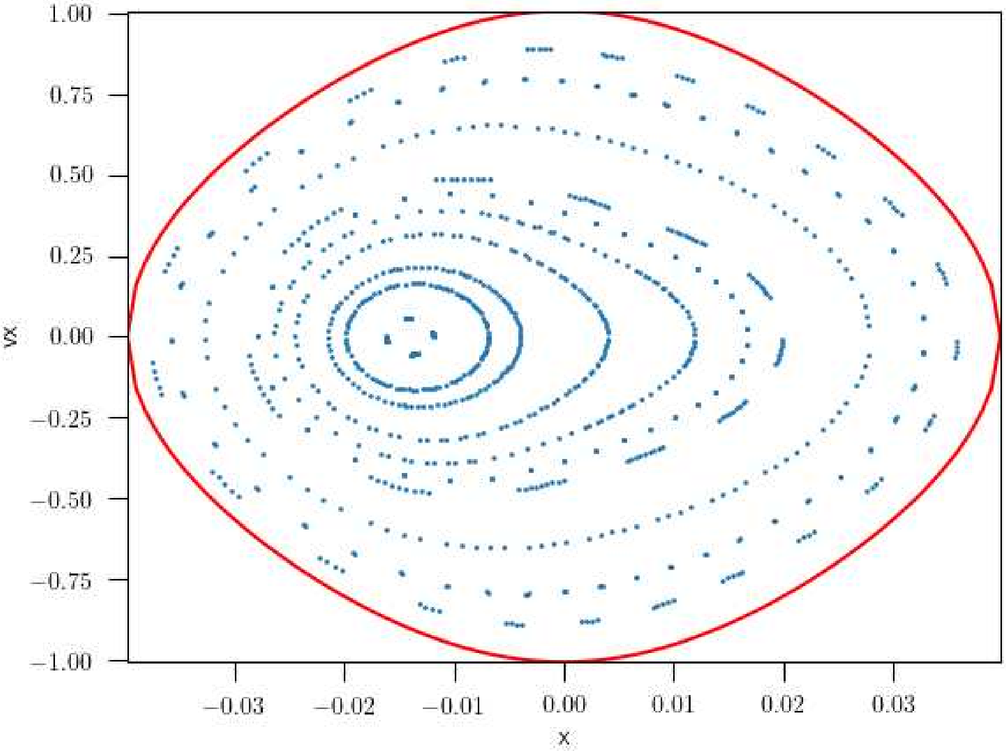
$\Omega = 1$ $\Omega = 0$



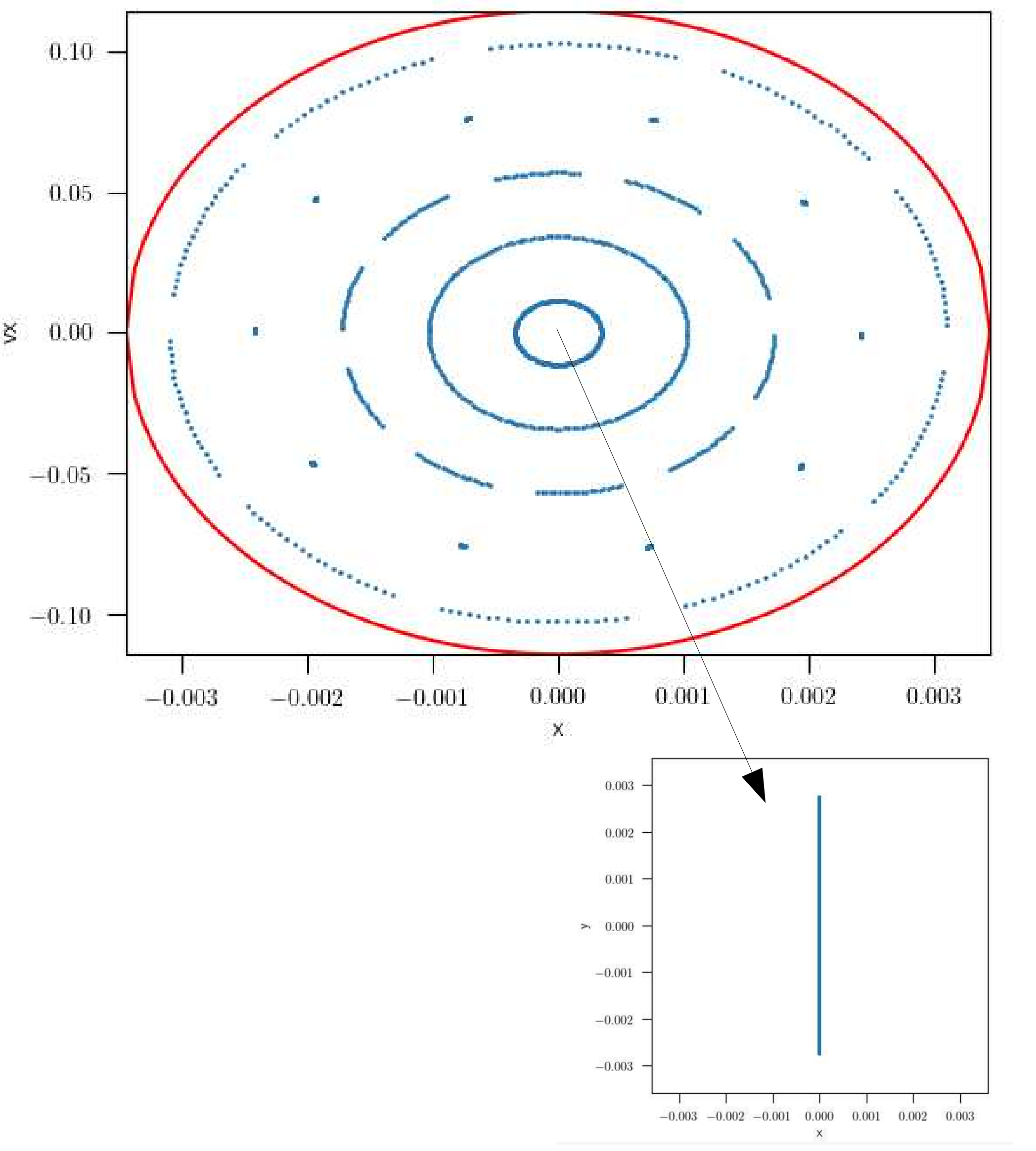

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Orbits around L_3

$$\Omega = 1$$



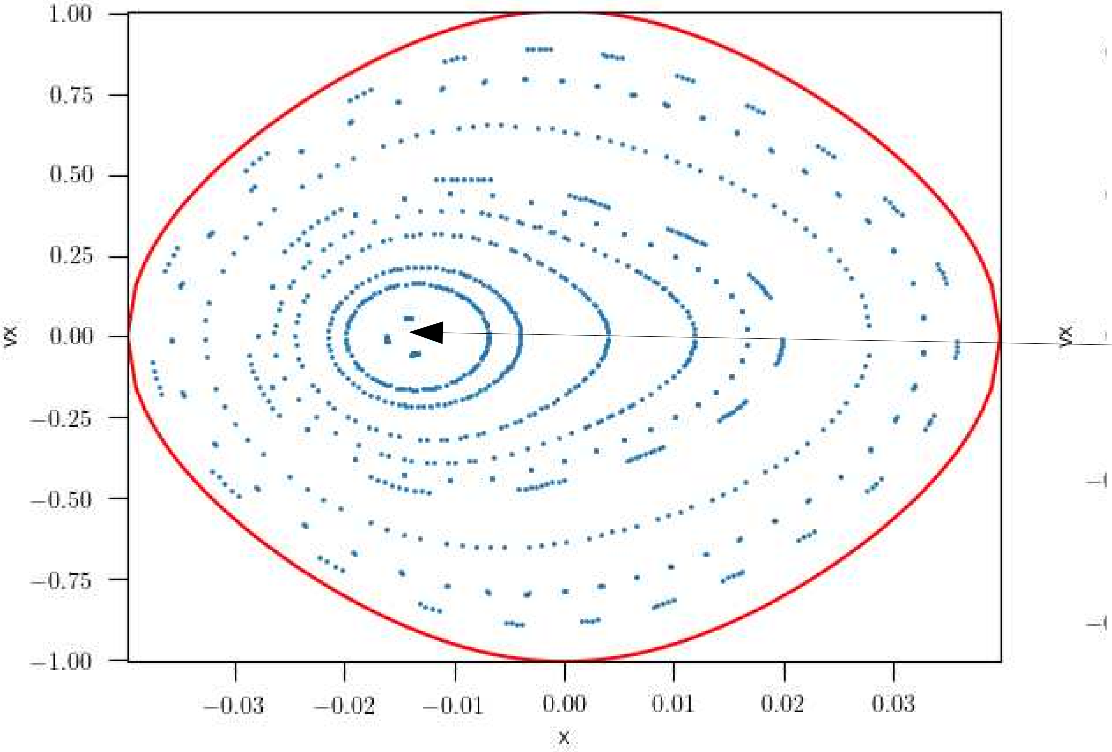
$$\Omega = 0$$



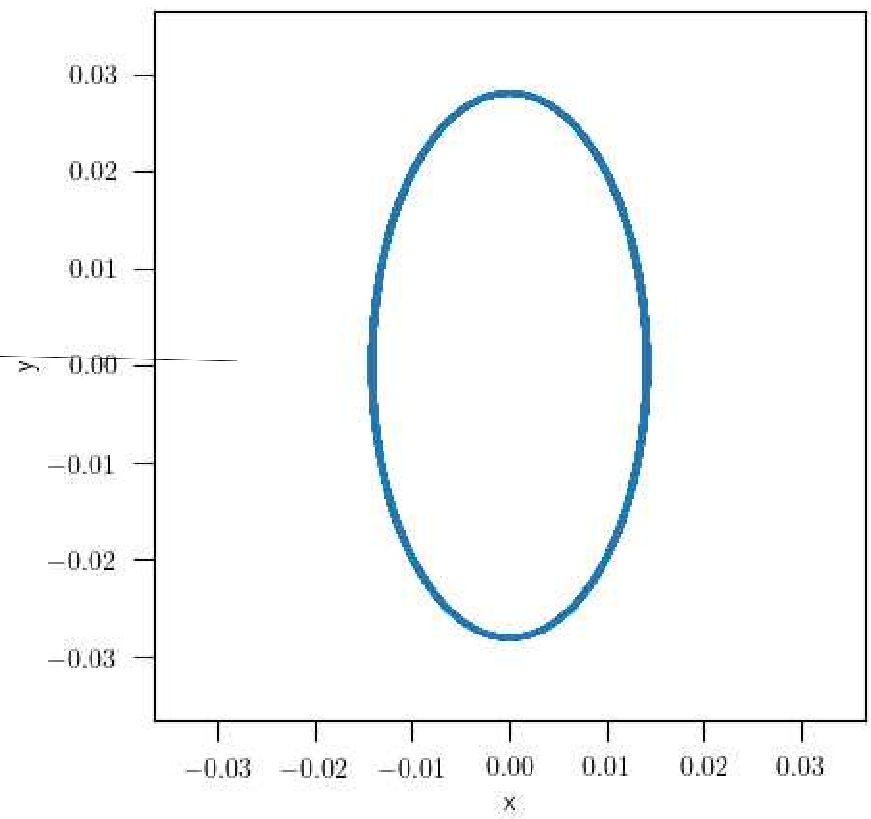

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Short axis (Y) orbits (periodic)

$\Omega = 1$



$\Omega = 0$



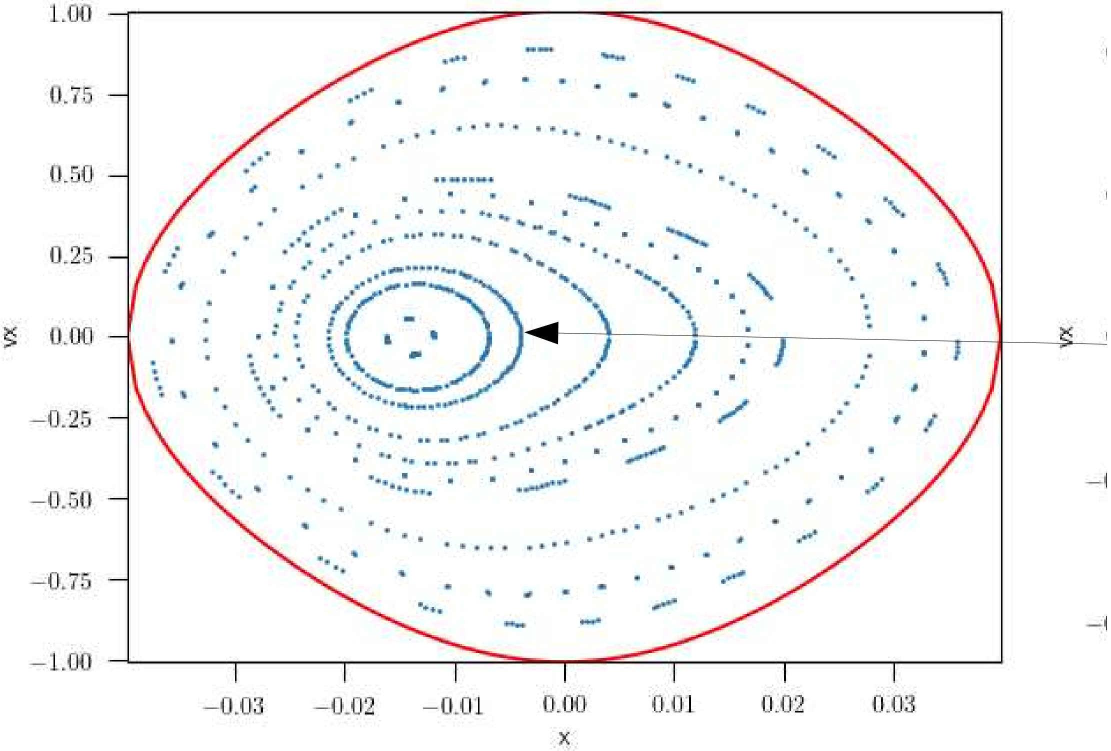
X4

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x -0.014
```

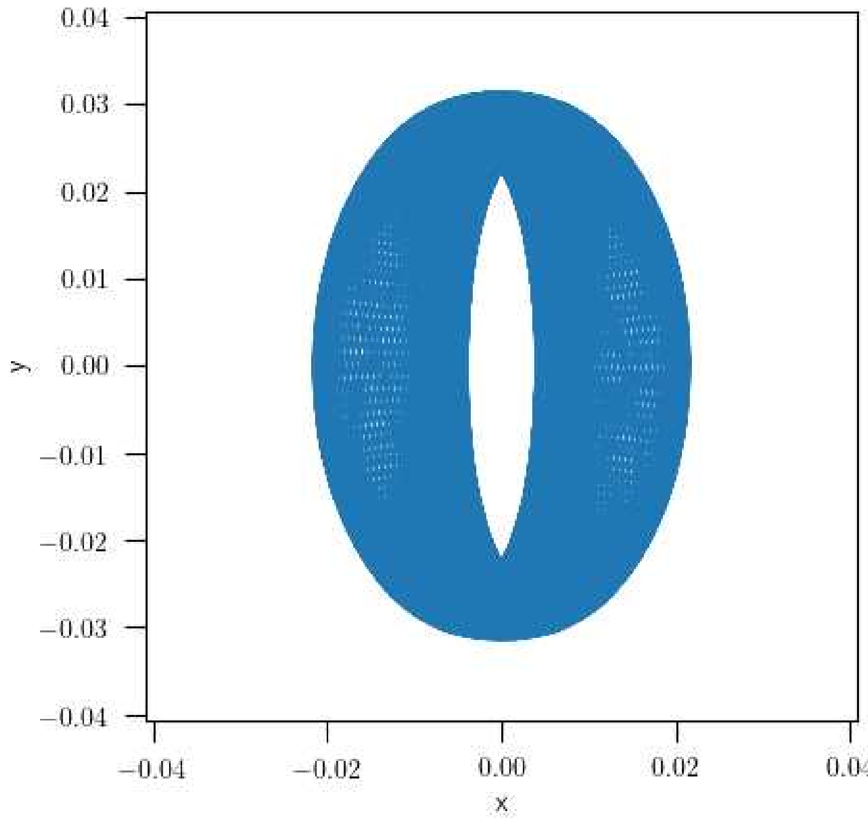
Apparition of a periodic loop orbit (replace the radial orbit, perpandicular to the bar), clockwise rotation

Short axis (Y) orbits (periodic)

$\Omega = 1$



$\Omega = 0$

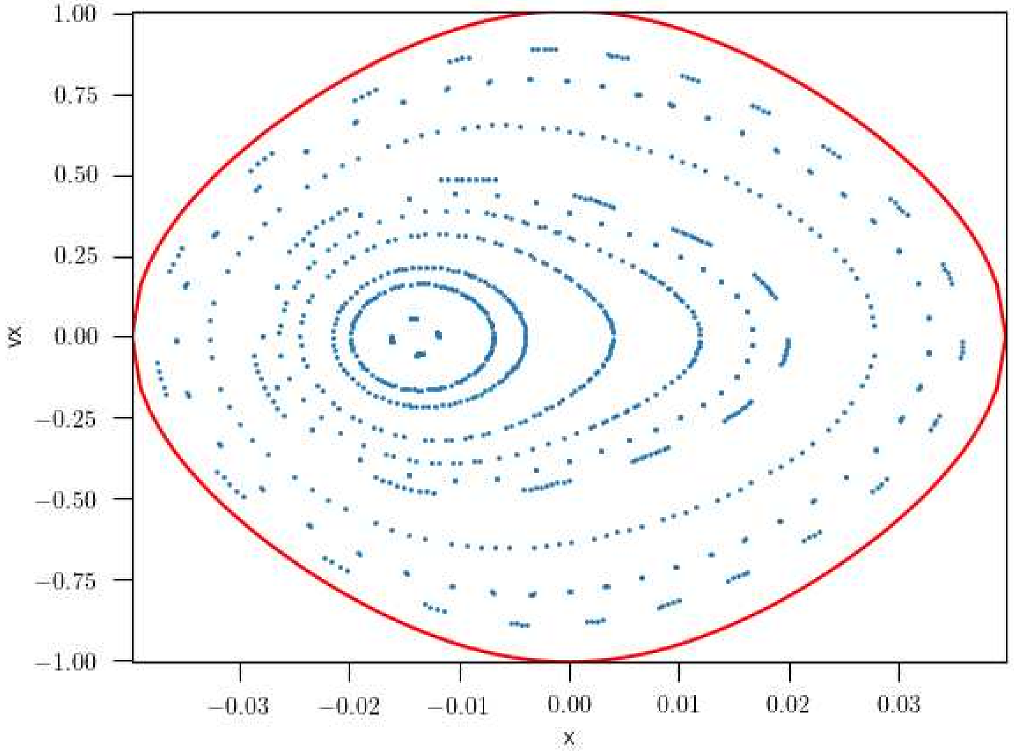


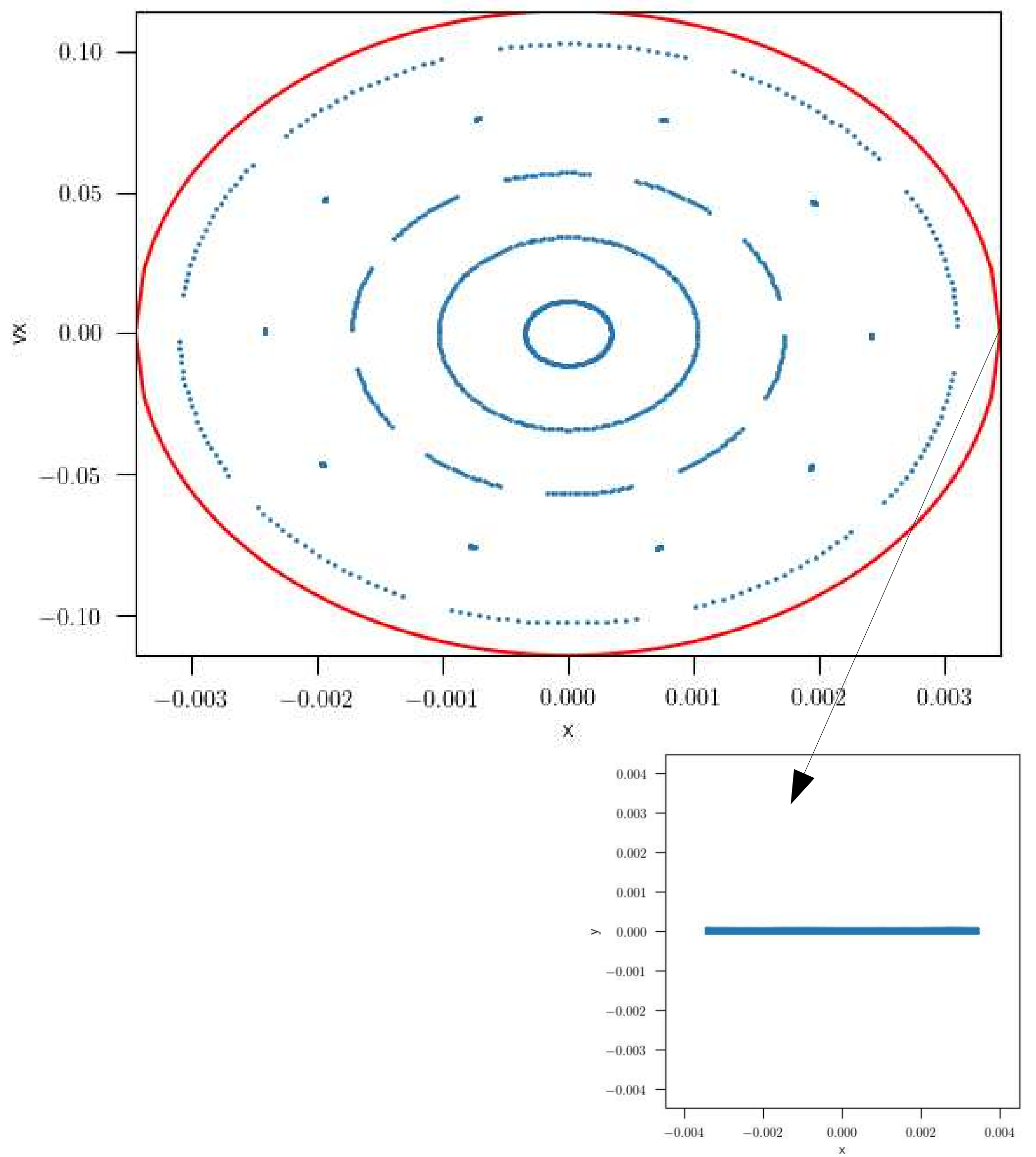
X4

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x -0.004
```

Orbits around L_3

$\Omega = 1$ $\Omega = 0$

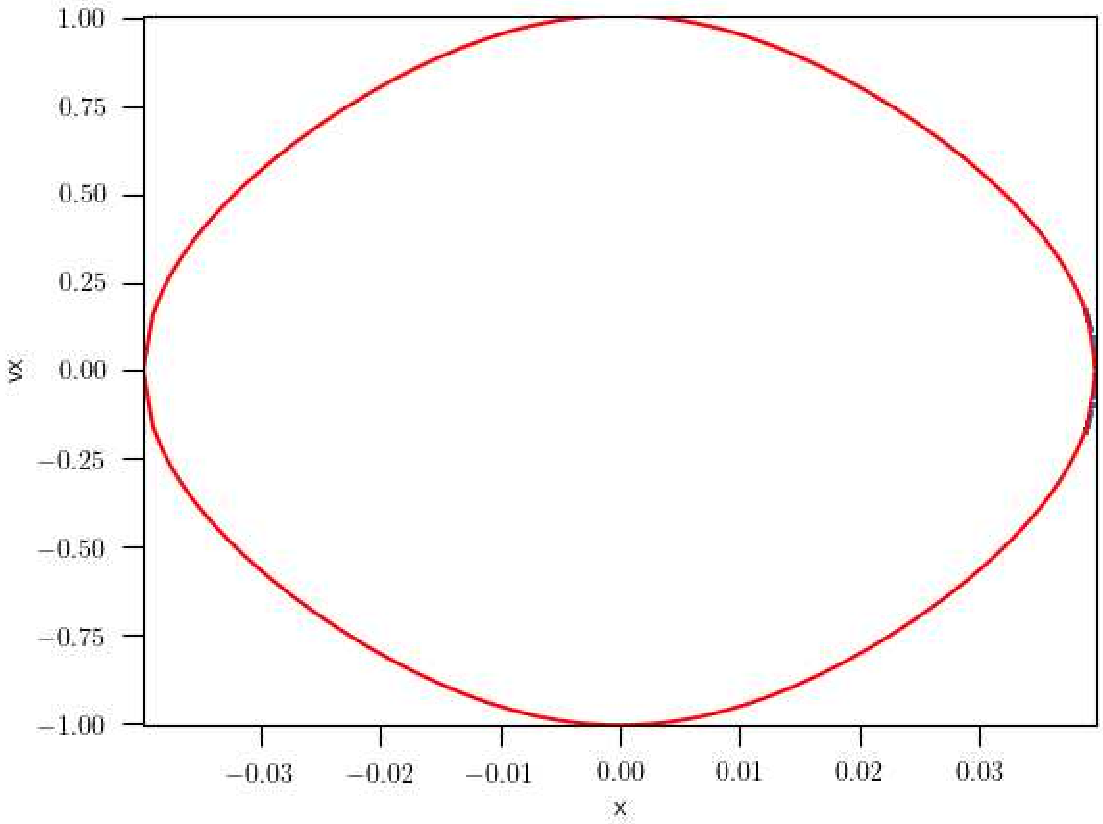


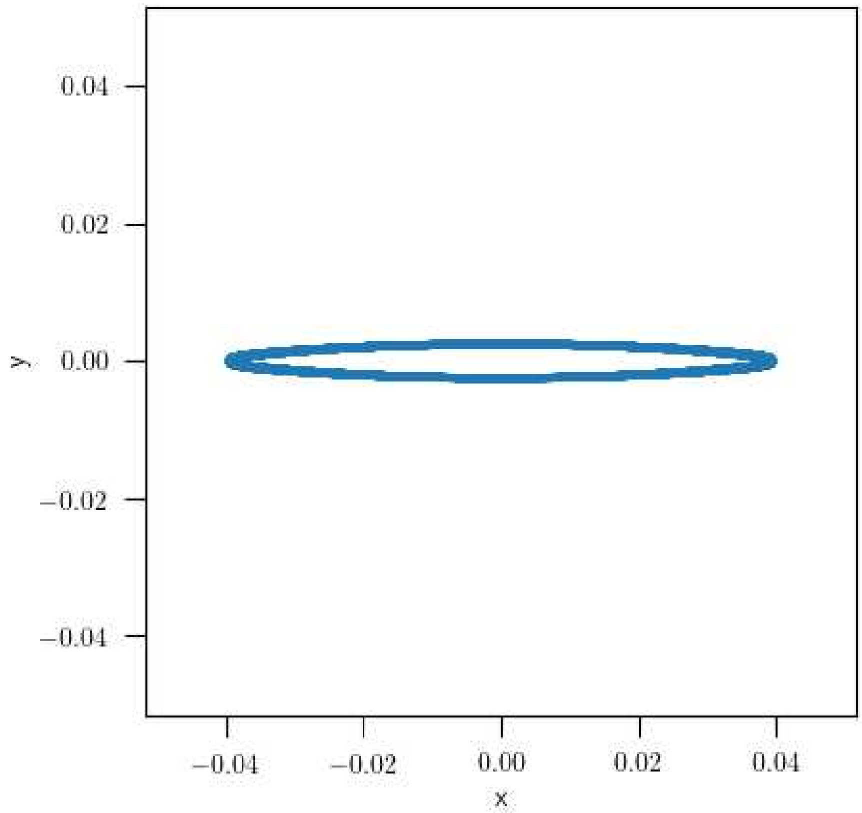



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 0 -E -3.5
```

Long axis (X) orbits (periodic)

$$\Omega = 1$$





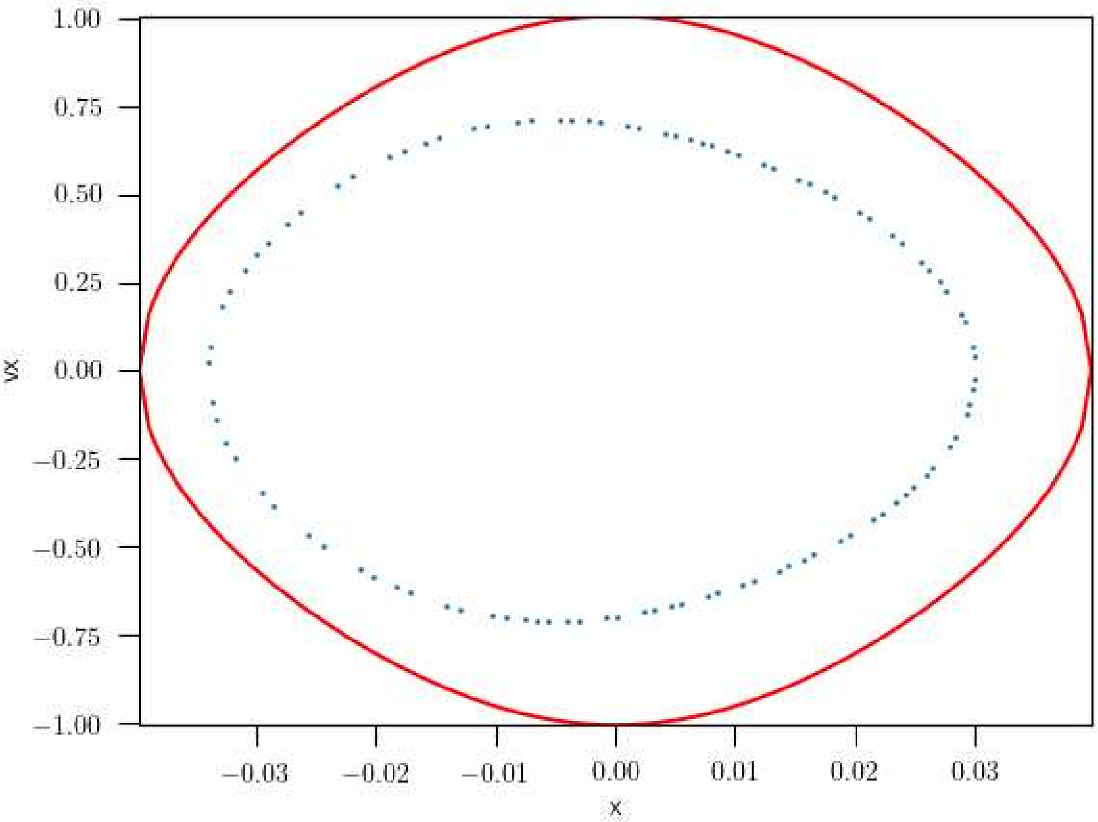
x1

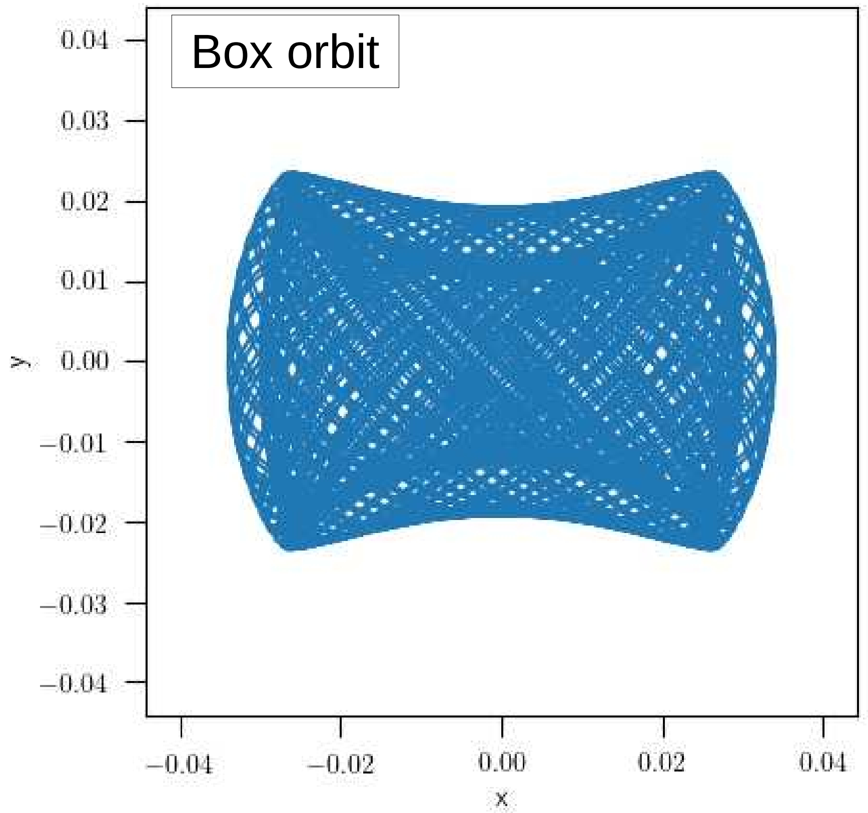
```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x 0.03975
```

Apparition of a periodic loop orbit (replace the radial orbit, parallel to the bar), anti-clockwise rotation

Long axis (X) orbits (non periodic)

$$\Omega = 1$$



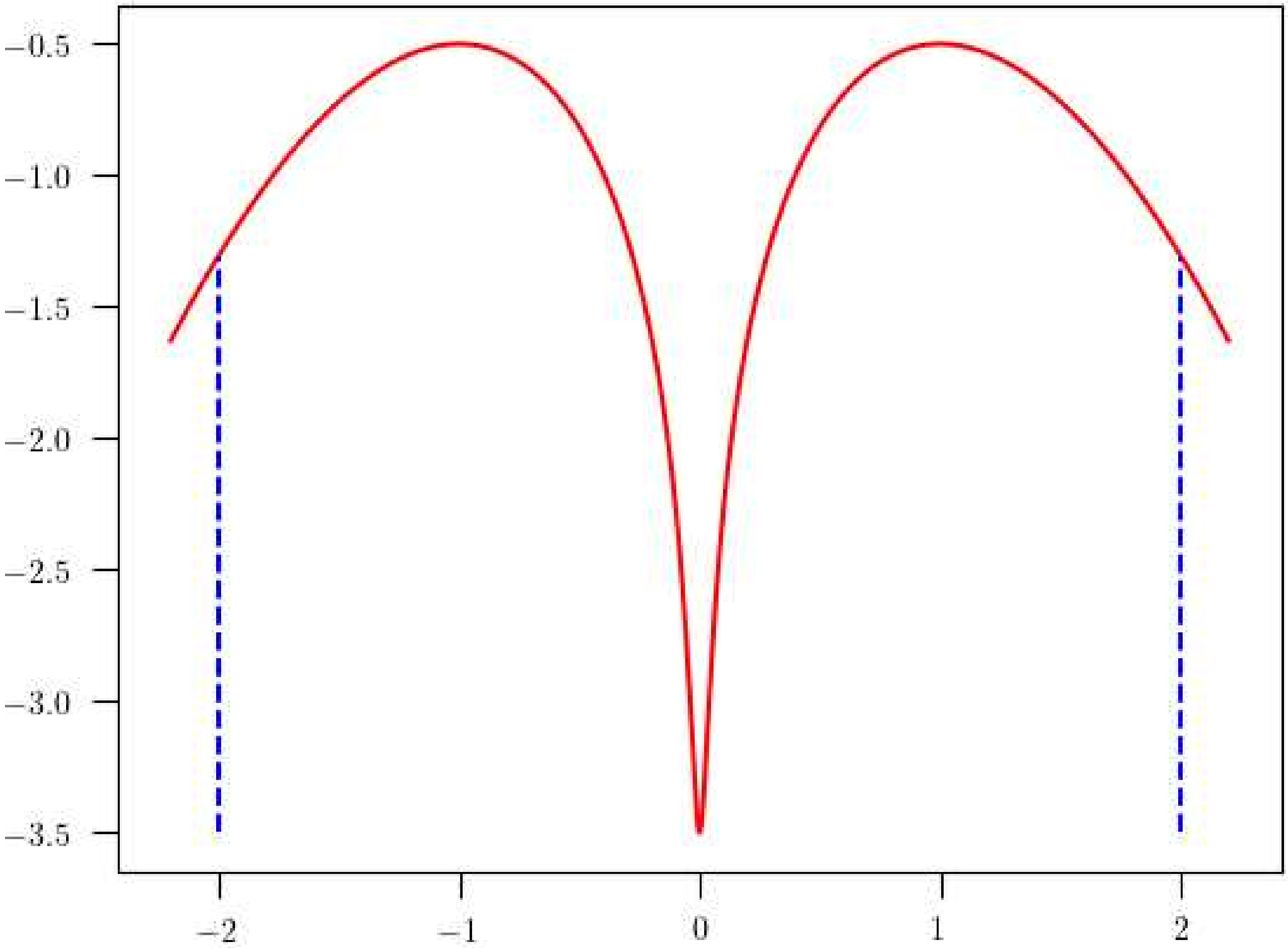


x1

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -3  --x 0.03
```

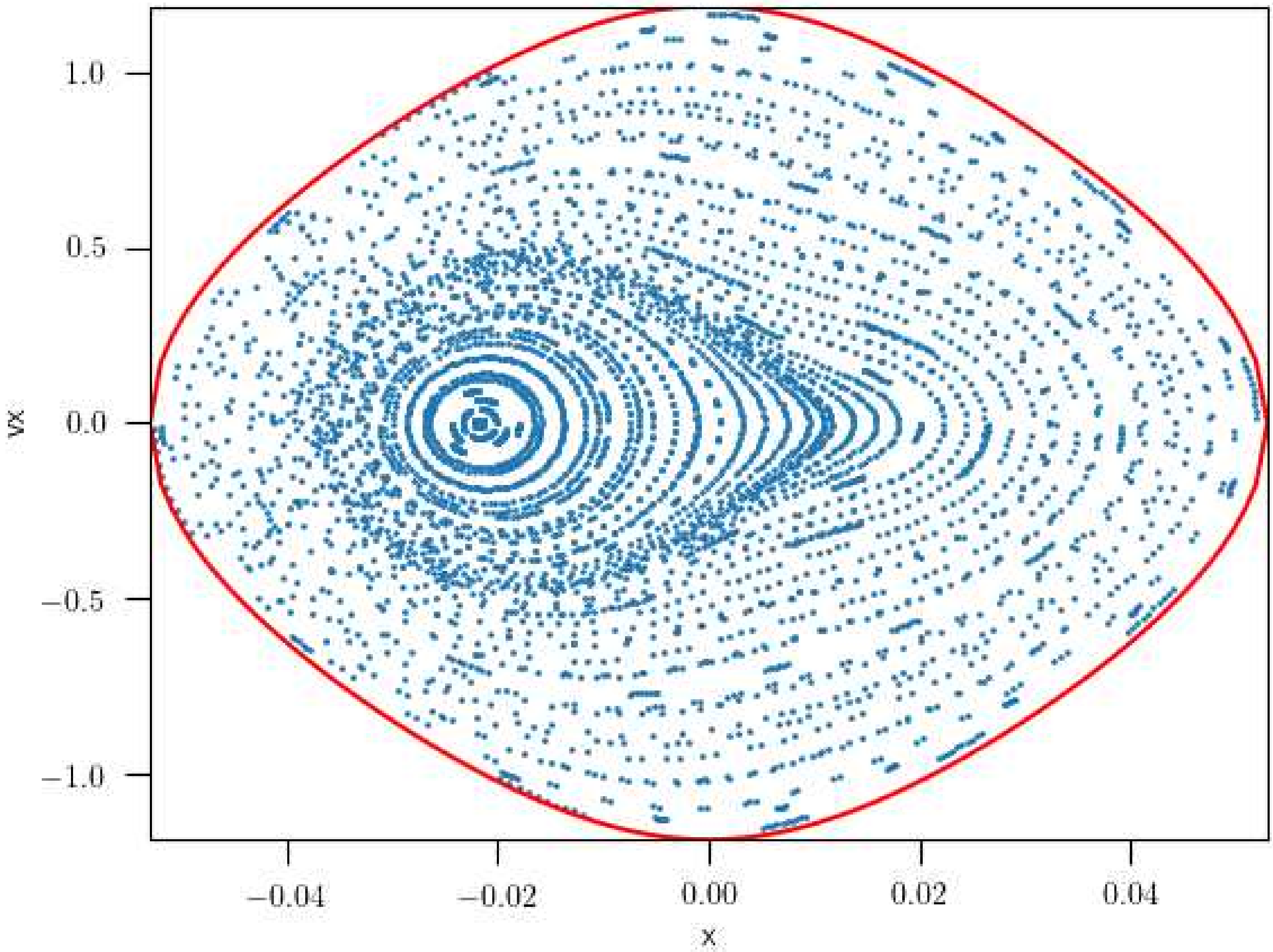
Increasing the energy

$$E = -2.8$$



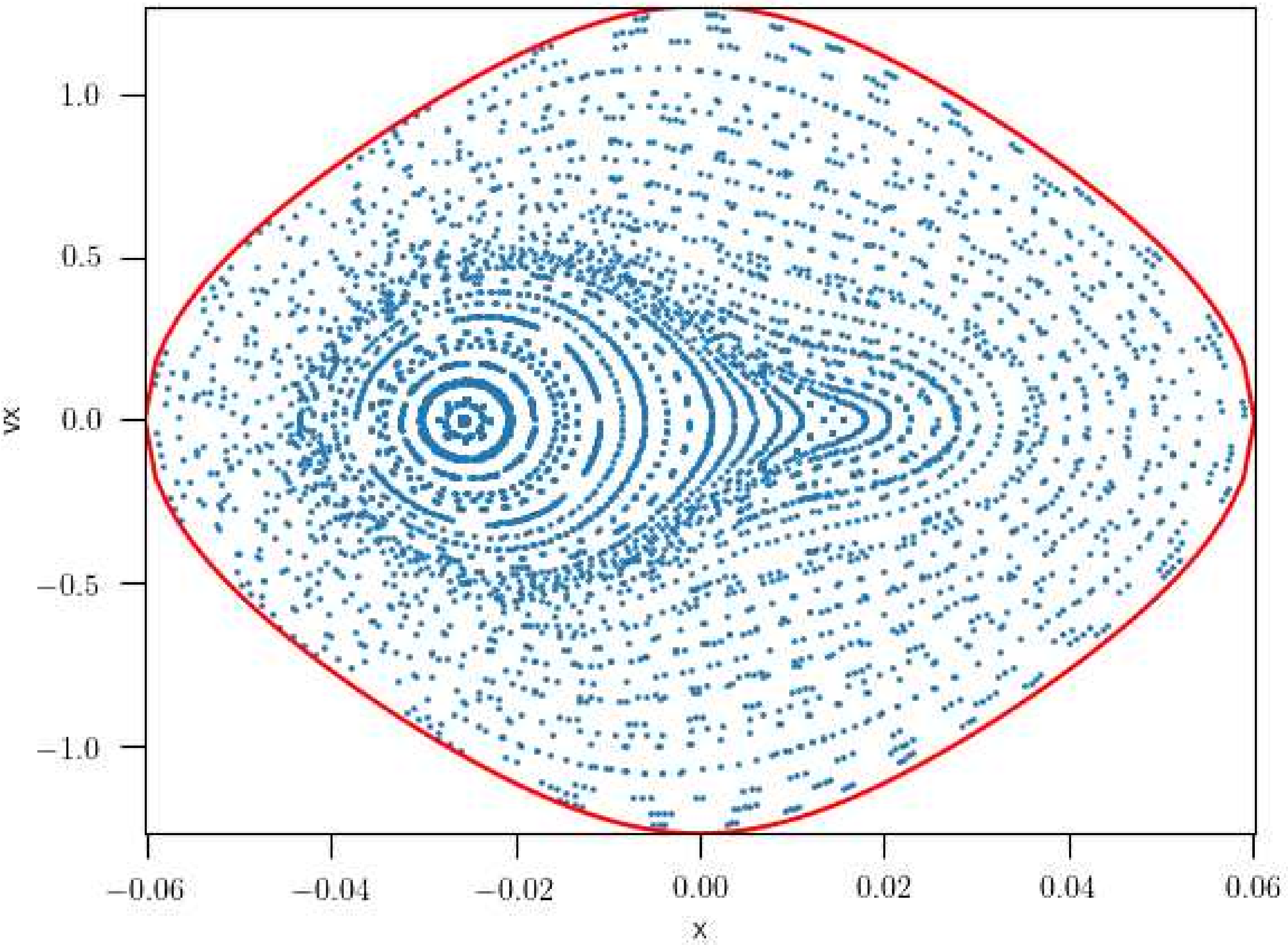

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

$$E = -2.8$$



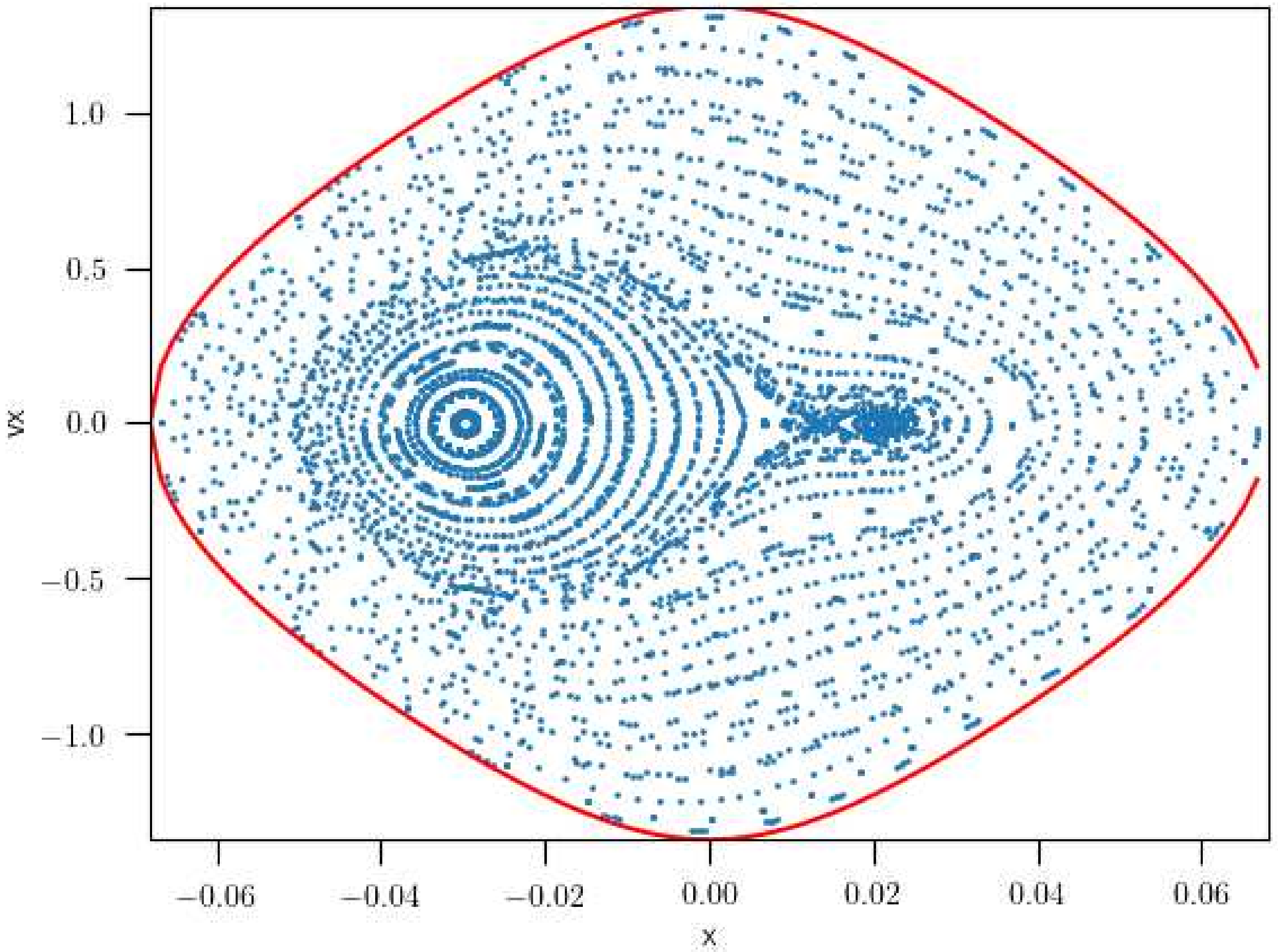

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.8 --norbits 50
```

$E = -2.7$



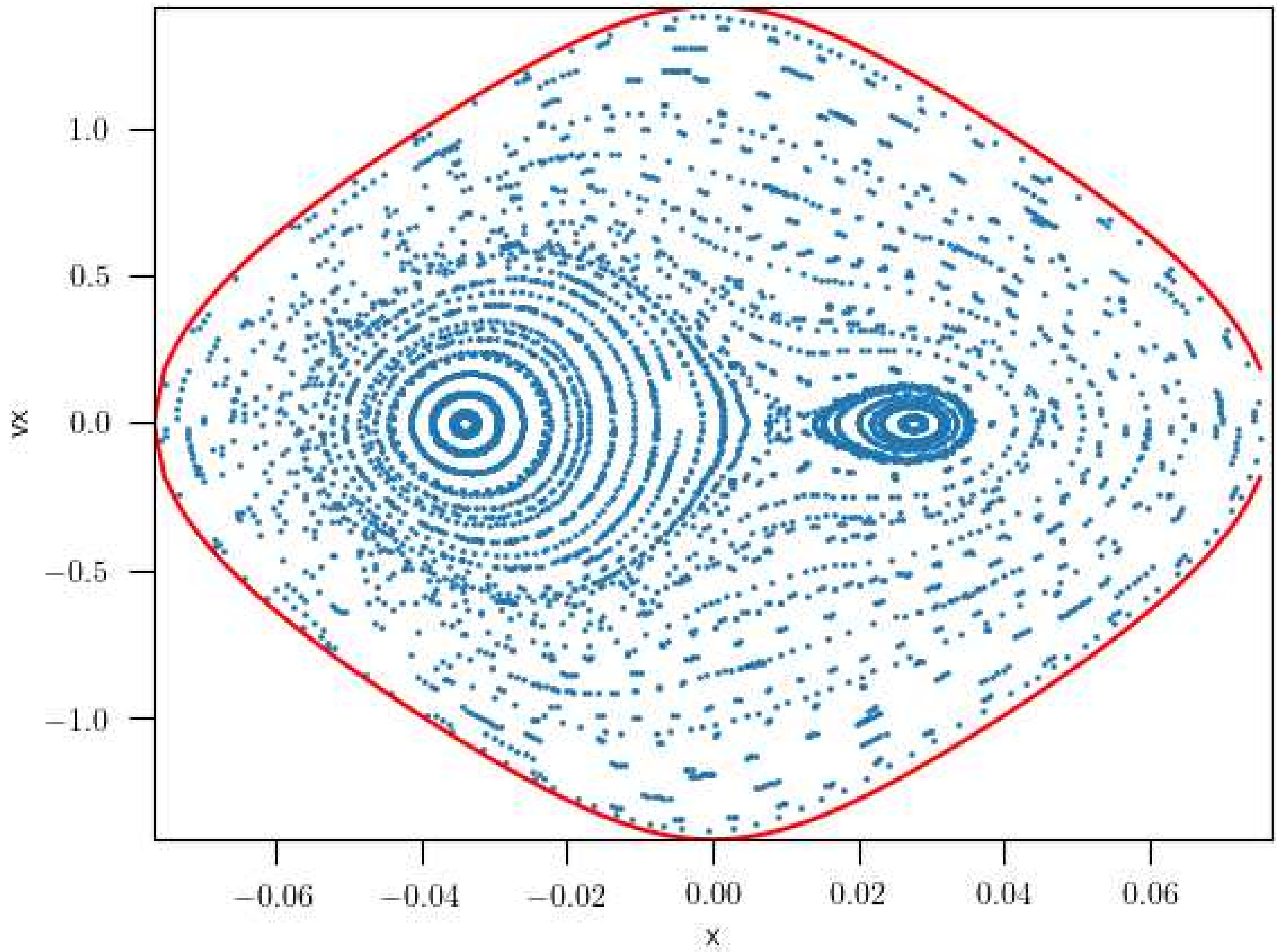

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.7 --norbits 50
```

$E = -2.6$



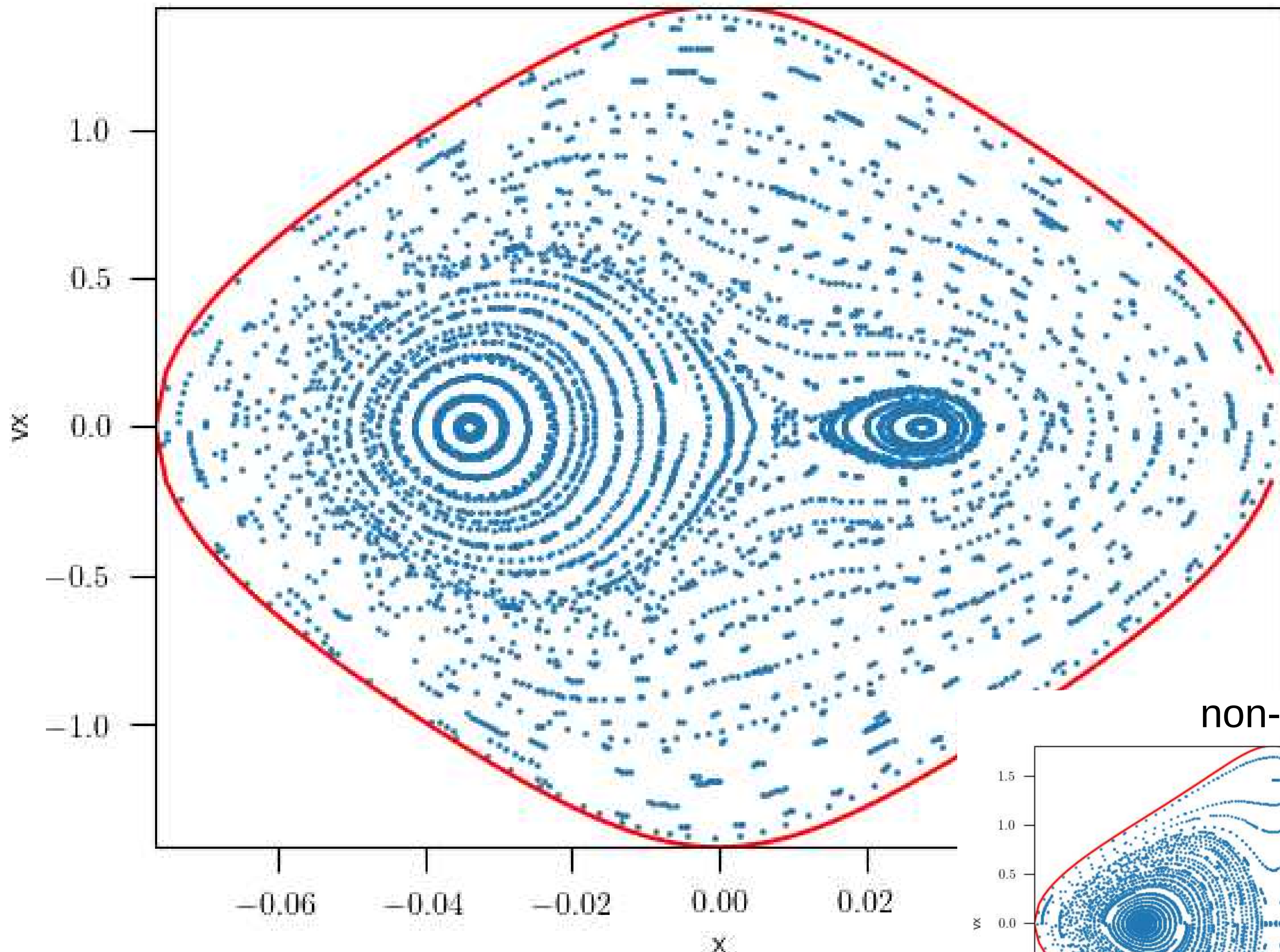

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.6 --norbits 50
```

$E = -2.5$

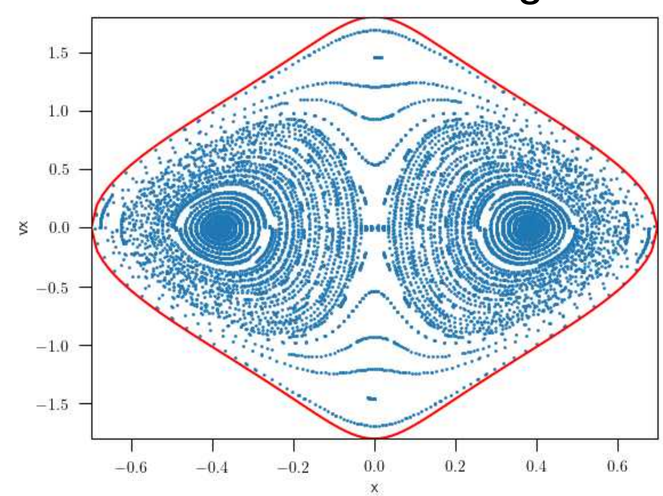



```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

$E = -2.5$

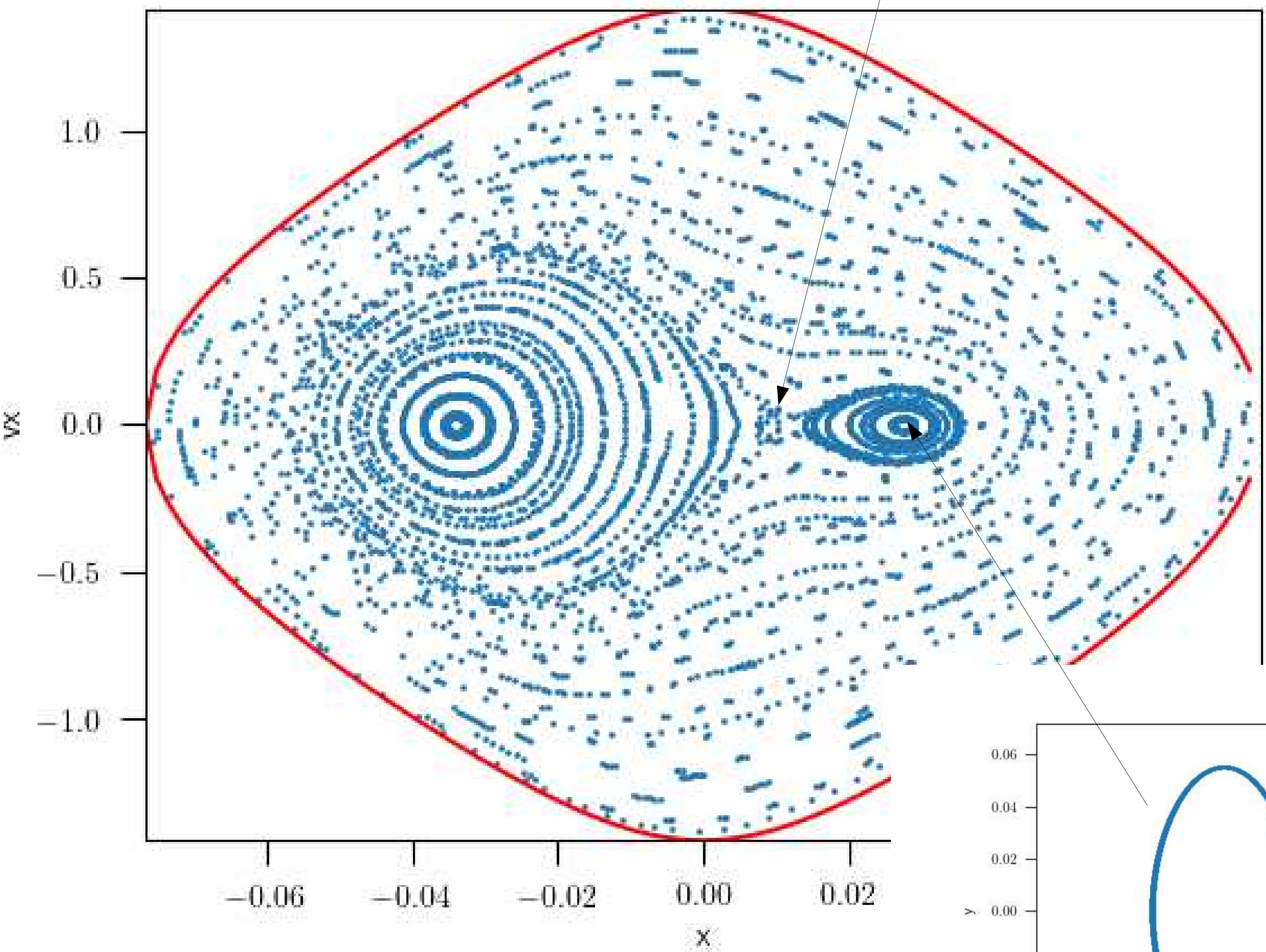


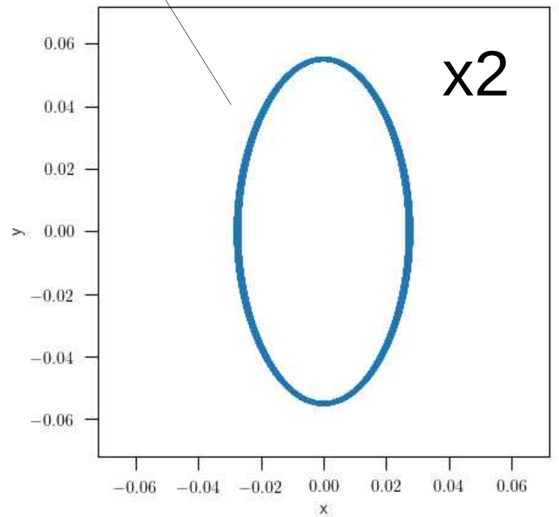
non-rotating case




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --norbits 50
```

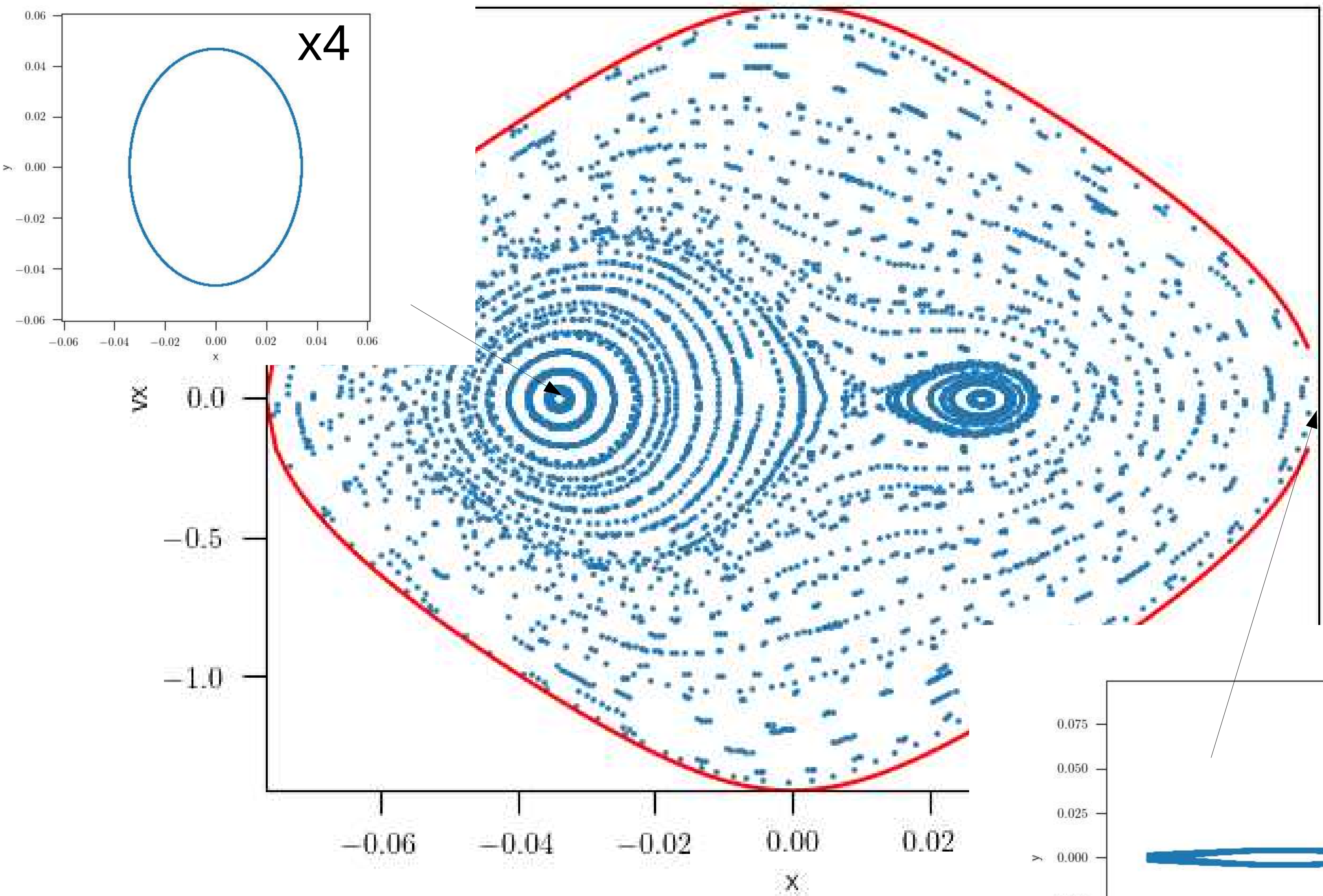
Bifurcation : apparition of x2 (stabe)/x3 (unstable) orbits

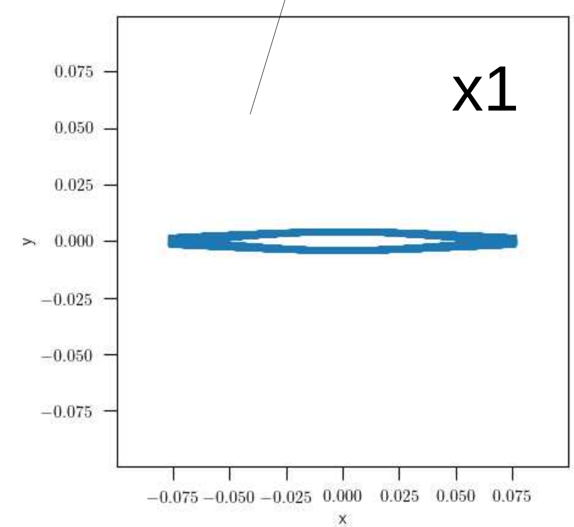





```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0268
```

x1 : prograde x4 : retrograde

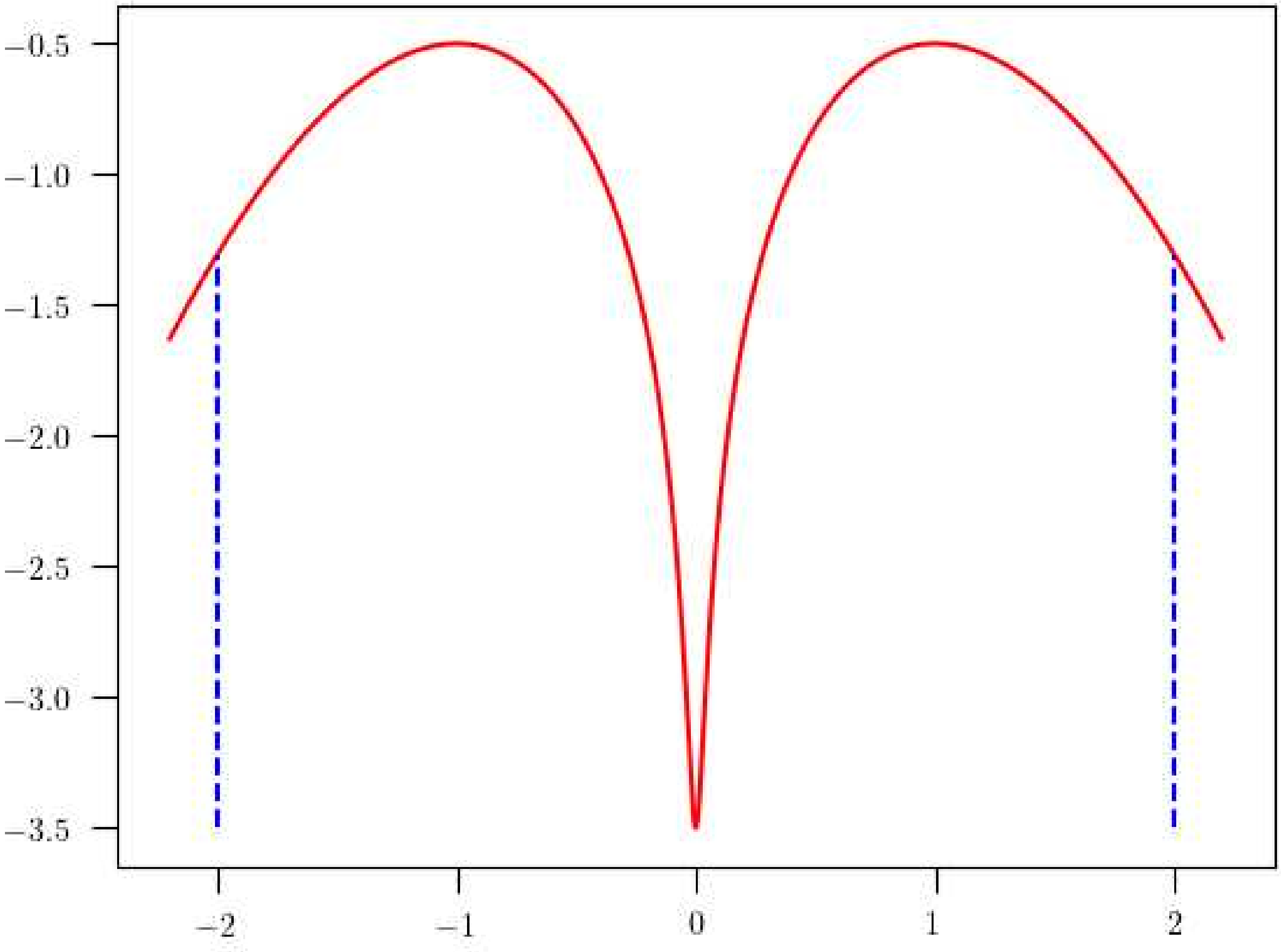





```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x -0.034
```

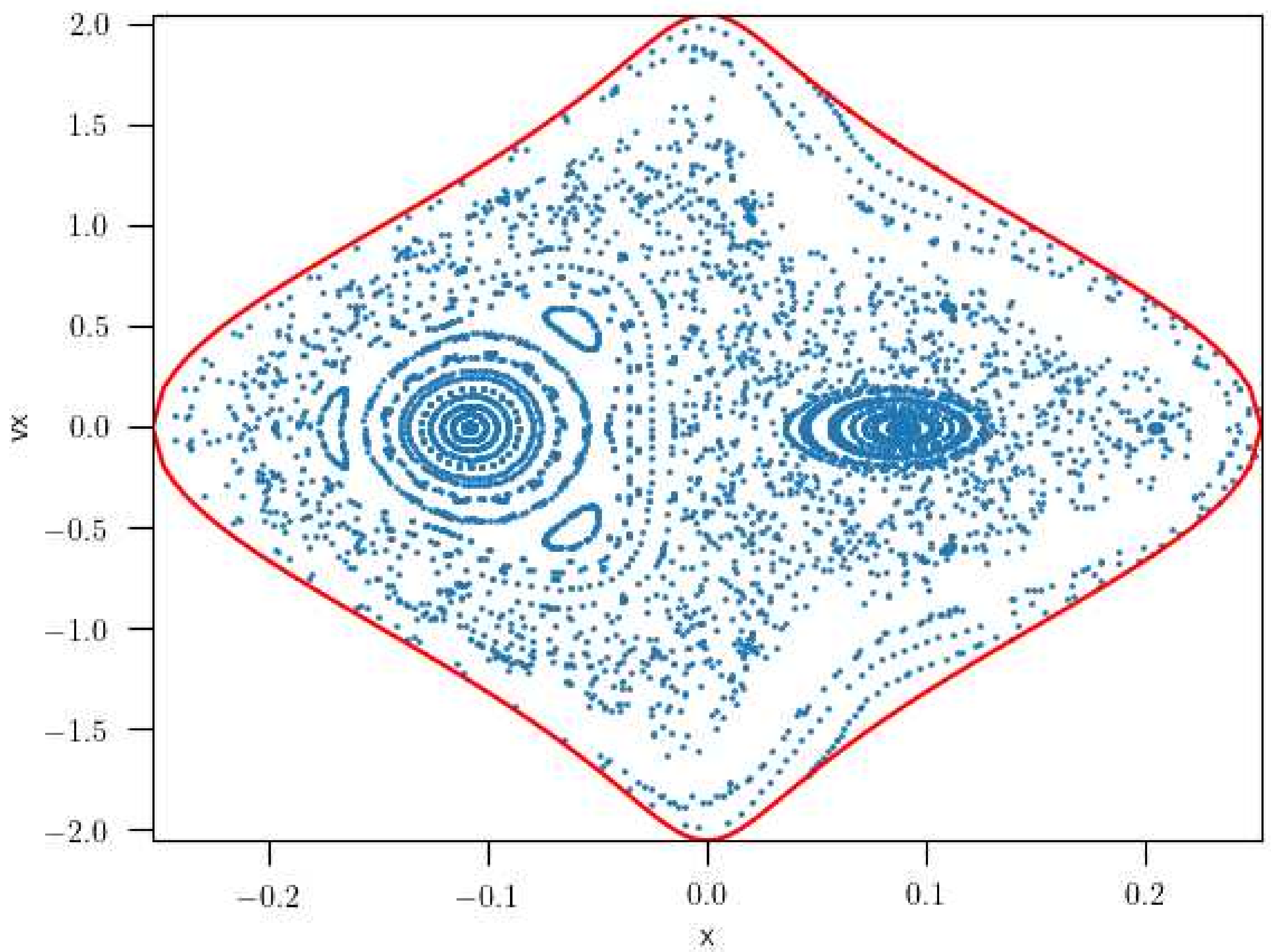
Increasing the energy

$$E = -1.4$$



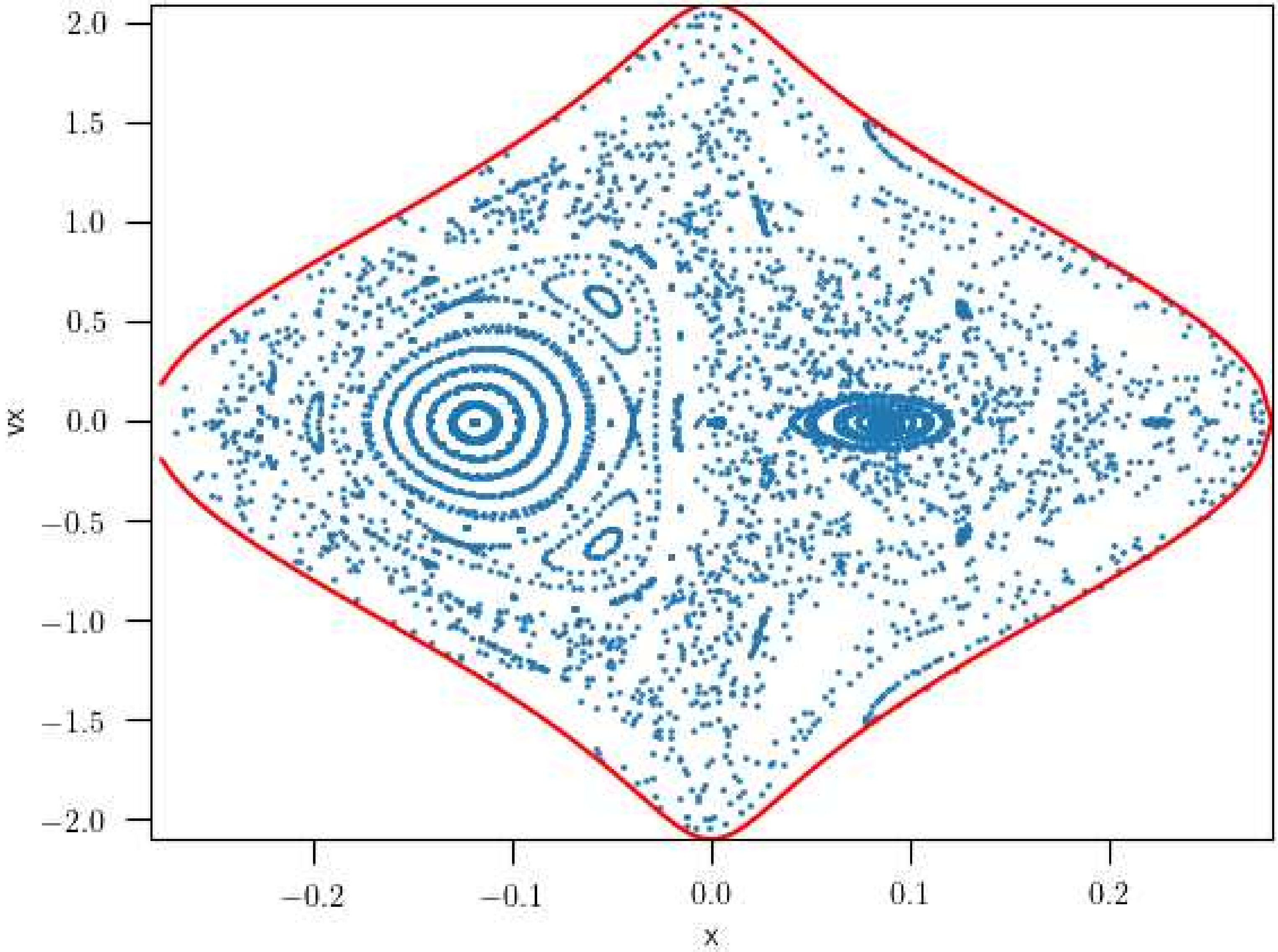

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 --plotpotential
```

$E = -1.4$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.4 --norbits 50
```

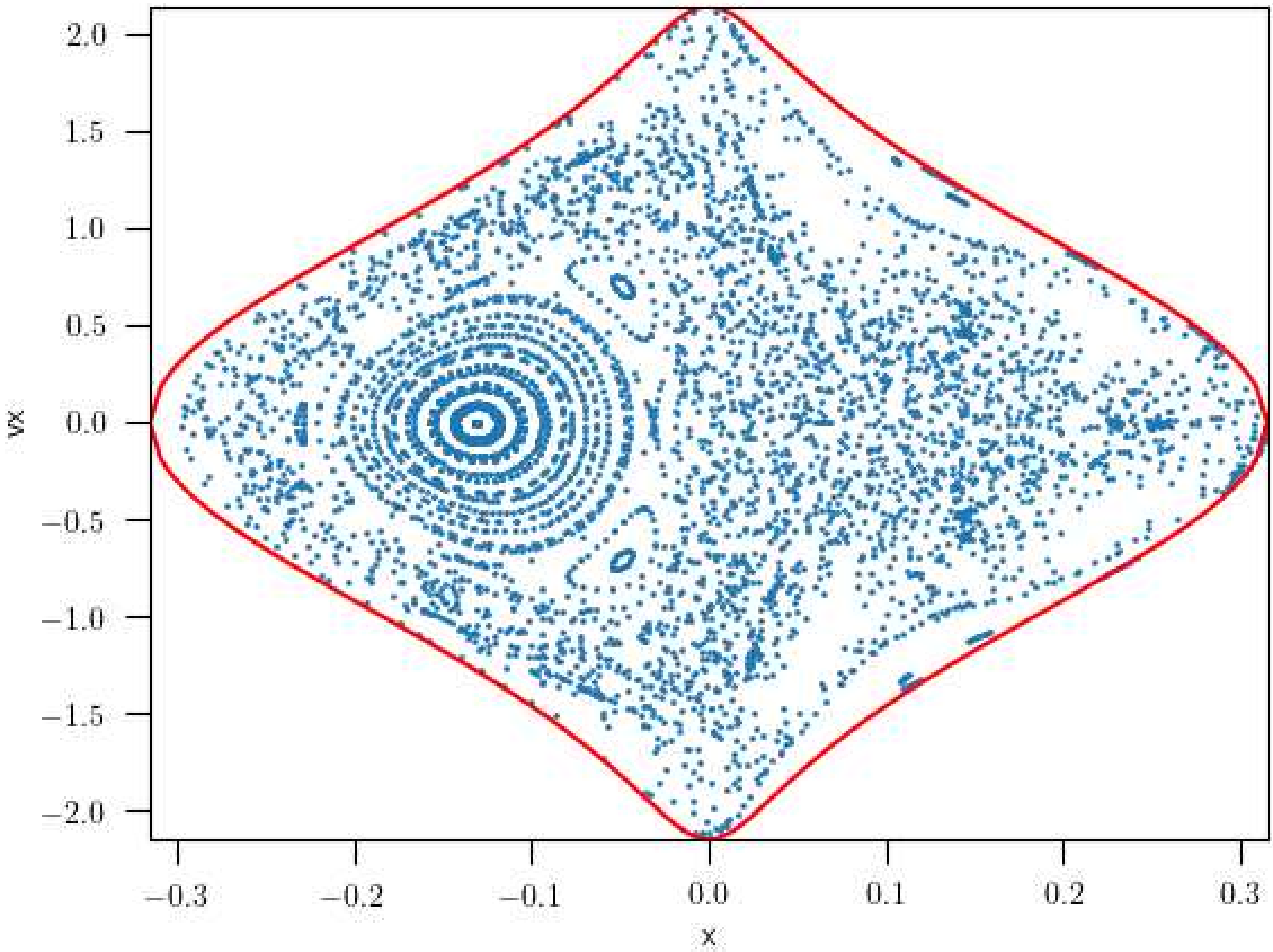
$E = -1.3$




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.3 --norbits 50
```

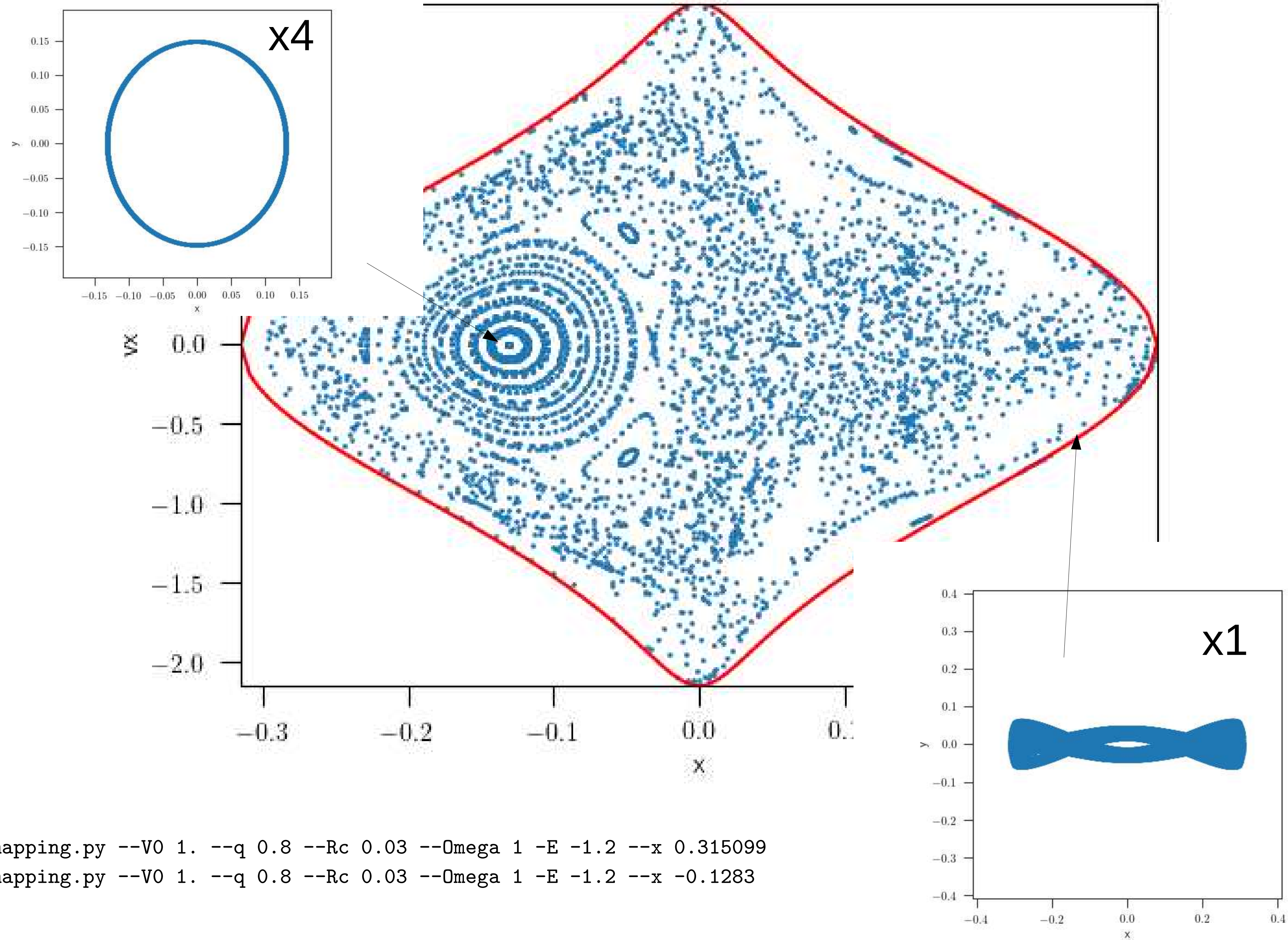
$E = -1.2$

Bifurcation : x2/x3 disappeared




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --norbits 50
```

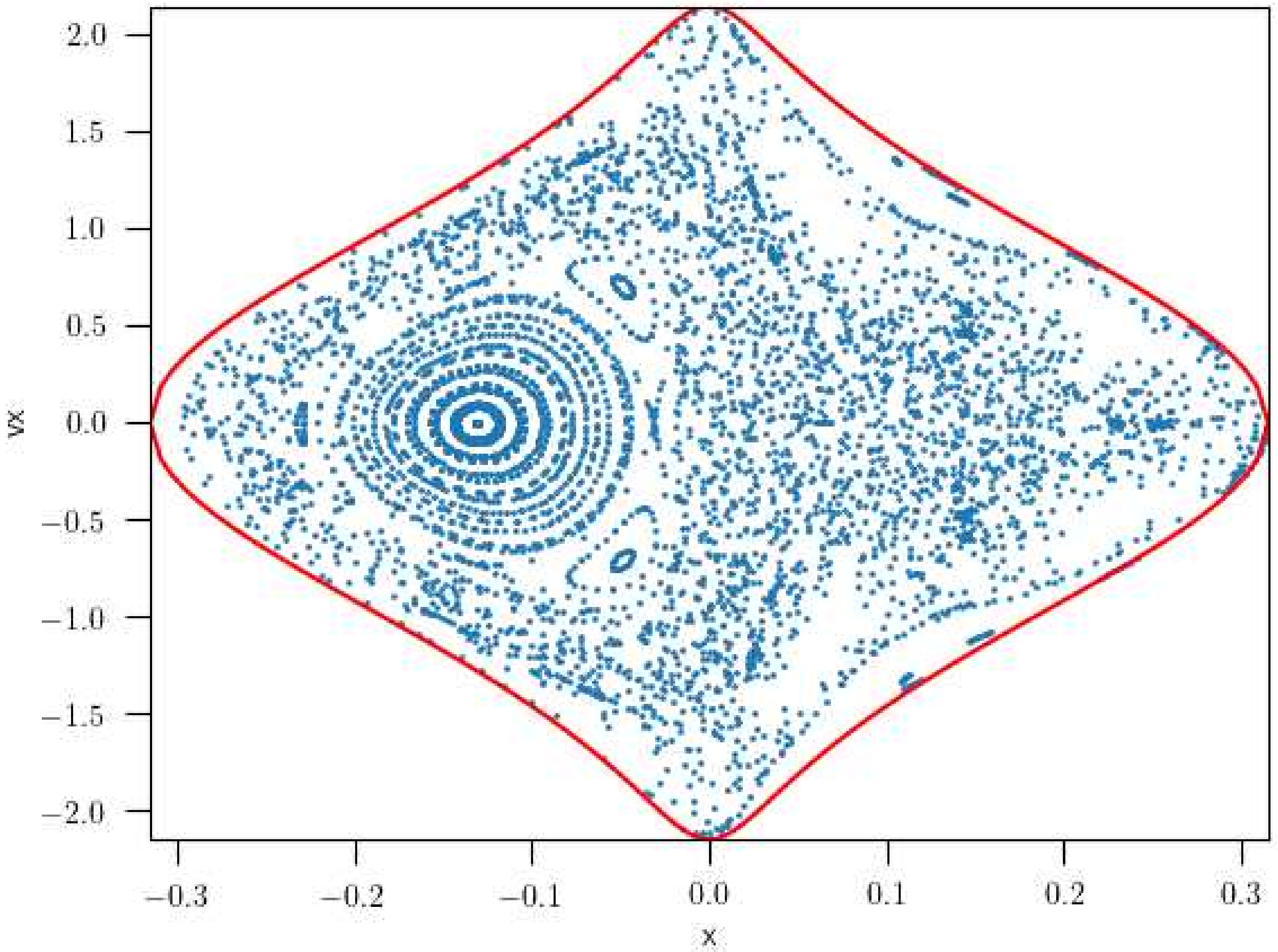
$$E = -1.2$$



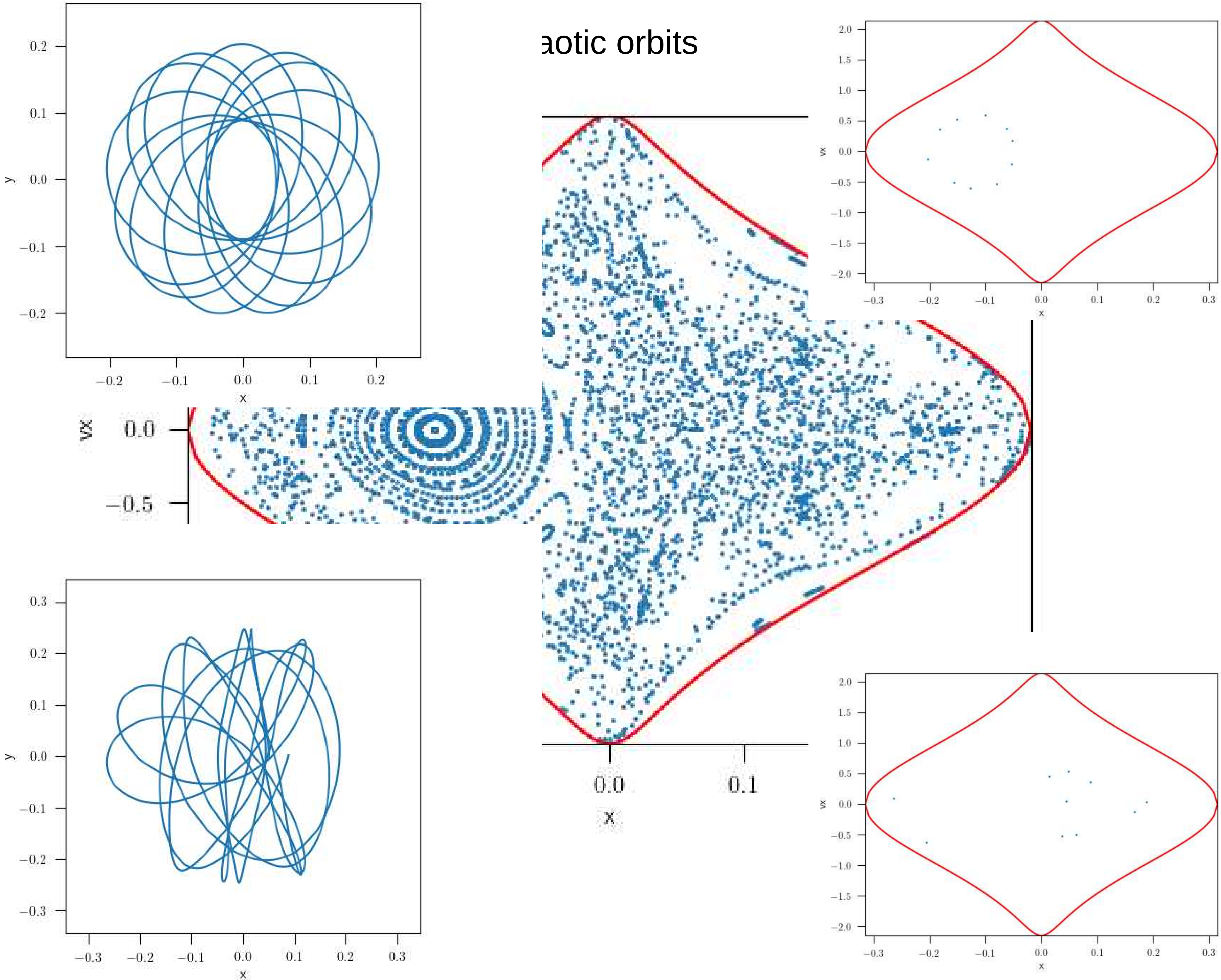

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x -0.1283
```

Chaotic orbits

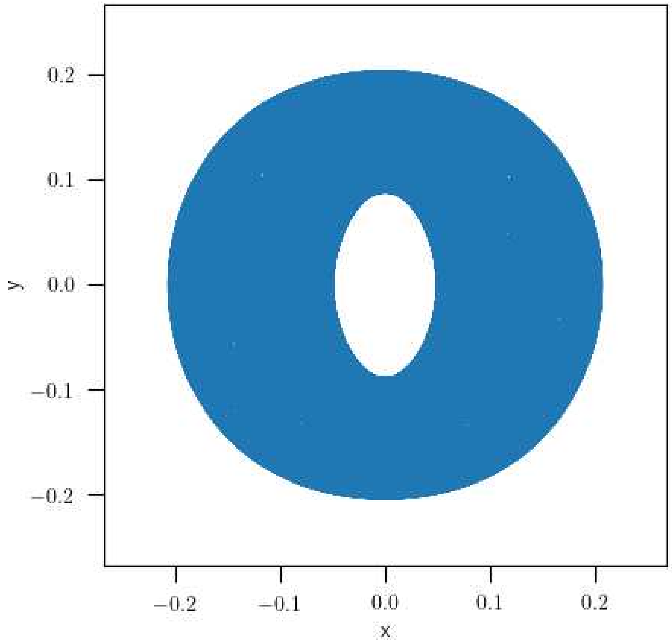
Chaotic orbits

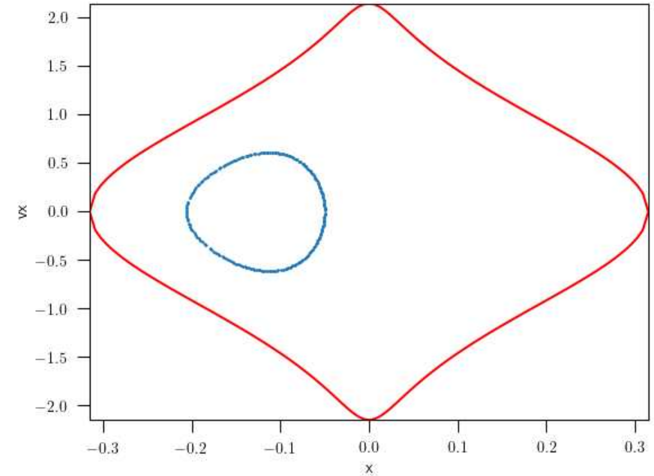


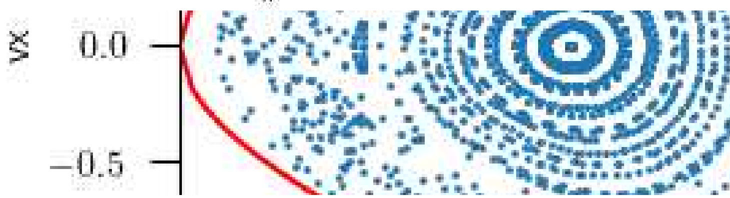
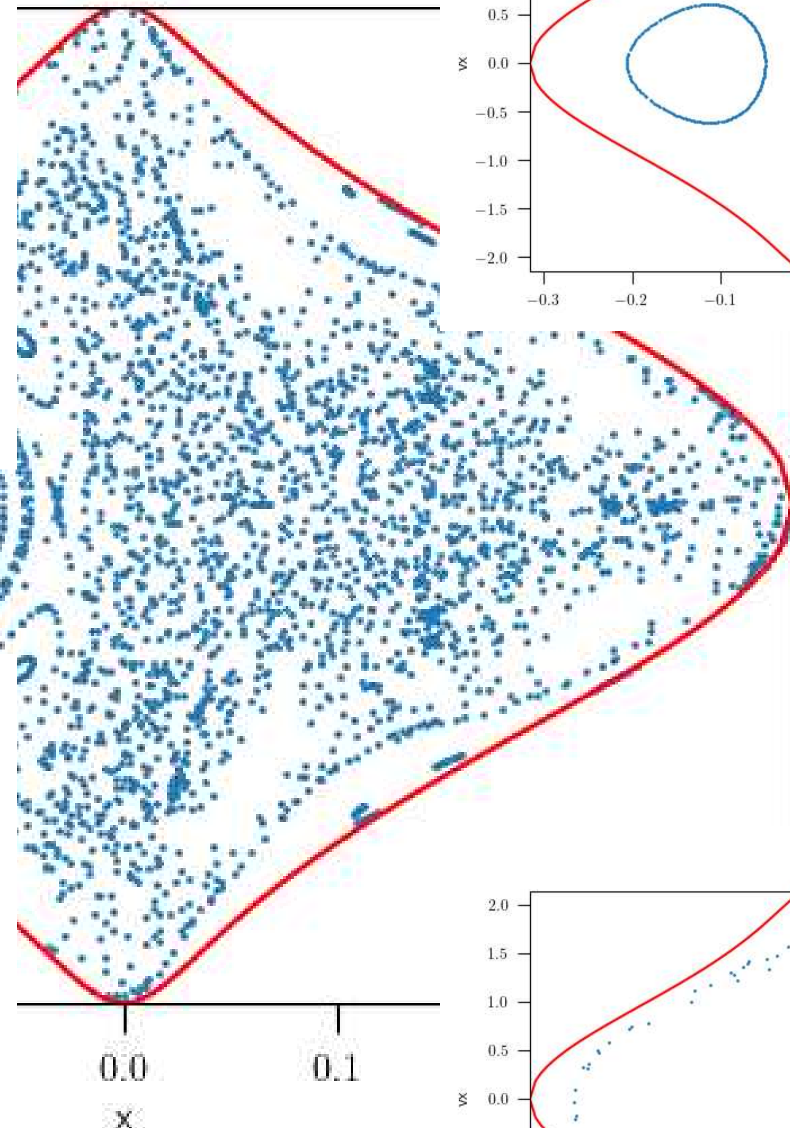
aotic orbits

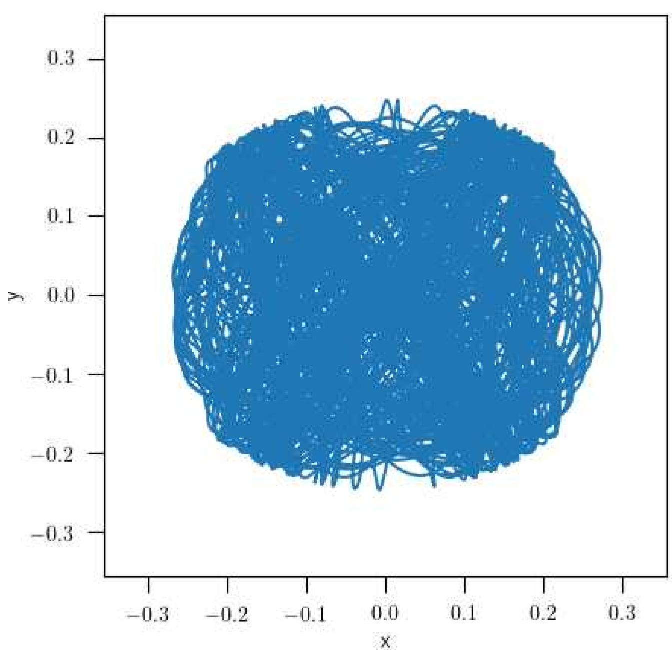


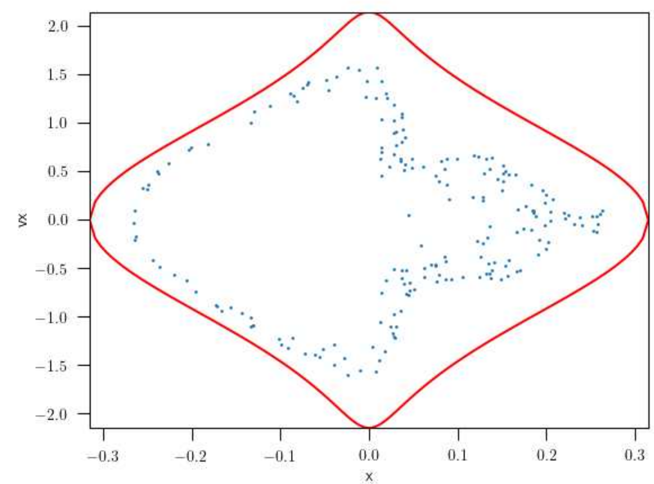
aotic orbits



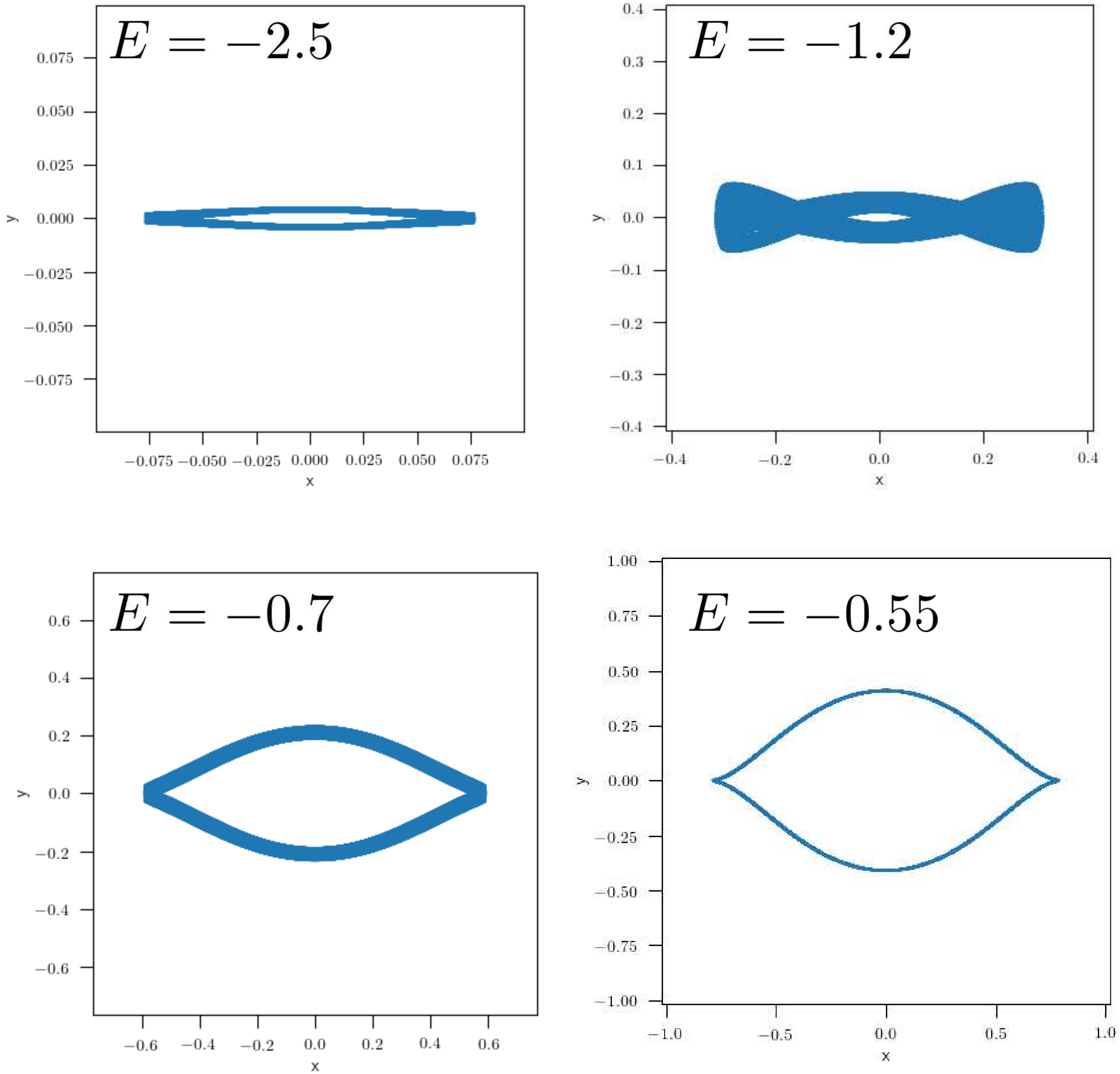






Evolution of the x1 orbit with increasing energy




```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5 --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -1.2 --x 0.315099
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.7 --x 0.590356
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -0.55 --x 0.783882
```

The X-orbit families
(characteristics curves)

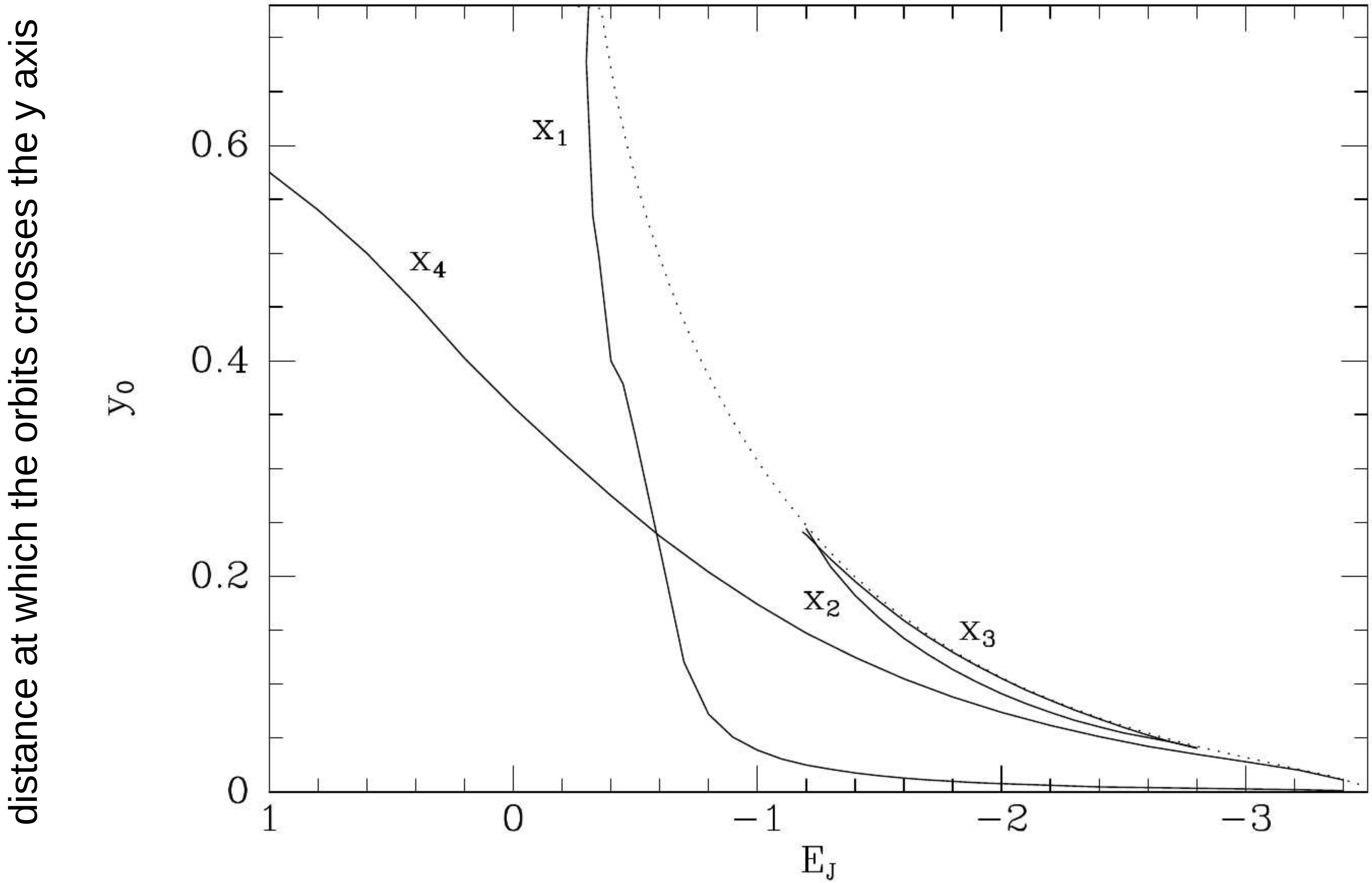


Figure 3.18 A plot of the Jacobi constant E_J of closed orbits in $\Phi_L(q = 0.8, R_c = 0.03, \Omega_b = 1)$ against the value of y at which the orbit cuts the potential's short axis. The dotted curve shows the relation $\Phi_{eff}(0, y) = E_J$. The families of orbits x_1–x_4 are marked.

Orbits
in weak rotating bars

Objective

- Split a loop orbit in two parts:

  - a circular motion of a guiding center

  - oscillations around the guiding center

Orbits in weak rotating bars (planar potentials)

- the barred potential rotates with a pattern speed Ω_b

Lagrangian :

$$\mathcal{L}(\vec{x},\dot{\vec{x}}) = \frac{1}{2}(\dot{\vec{x}} + \vec{\Omega}_b \times \vec{x})^2 - \phi(\vec{x})$$

In 2-D , with $\vec{\Omega}_b = \Omega_b \vec{e}_z$

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x} - y\,\Omega_b)^2 + \frac{1}{2}(\dot{y} + x\,\Omega_b)^2 - \phi(x,y)$$

In cylindrical coordinates

$$\mathcal{L}(R, \varphi, \dot{R}, \dot{\varphi}) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\left(R(\dot{\varphi} + \Omega_b)\right)^2 - \phi(R,\varphi)$$

Equations of motion in cylindrical coordinates (Euler-Lagrange)

$$\begin{cases} \ddot{R} = R(\dot{\varphi} + \Omega_b)^2 - \dfrac{\partial \phi}{\partial R} \\ \dfrac{d}{dt}\left(R^2(\dot{\varphi} + \Omega_b)\right) = - \dfrac{\partial \phi}{\partial \varphi} \end{cases}$$

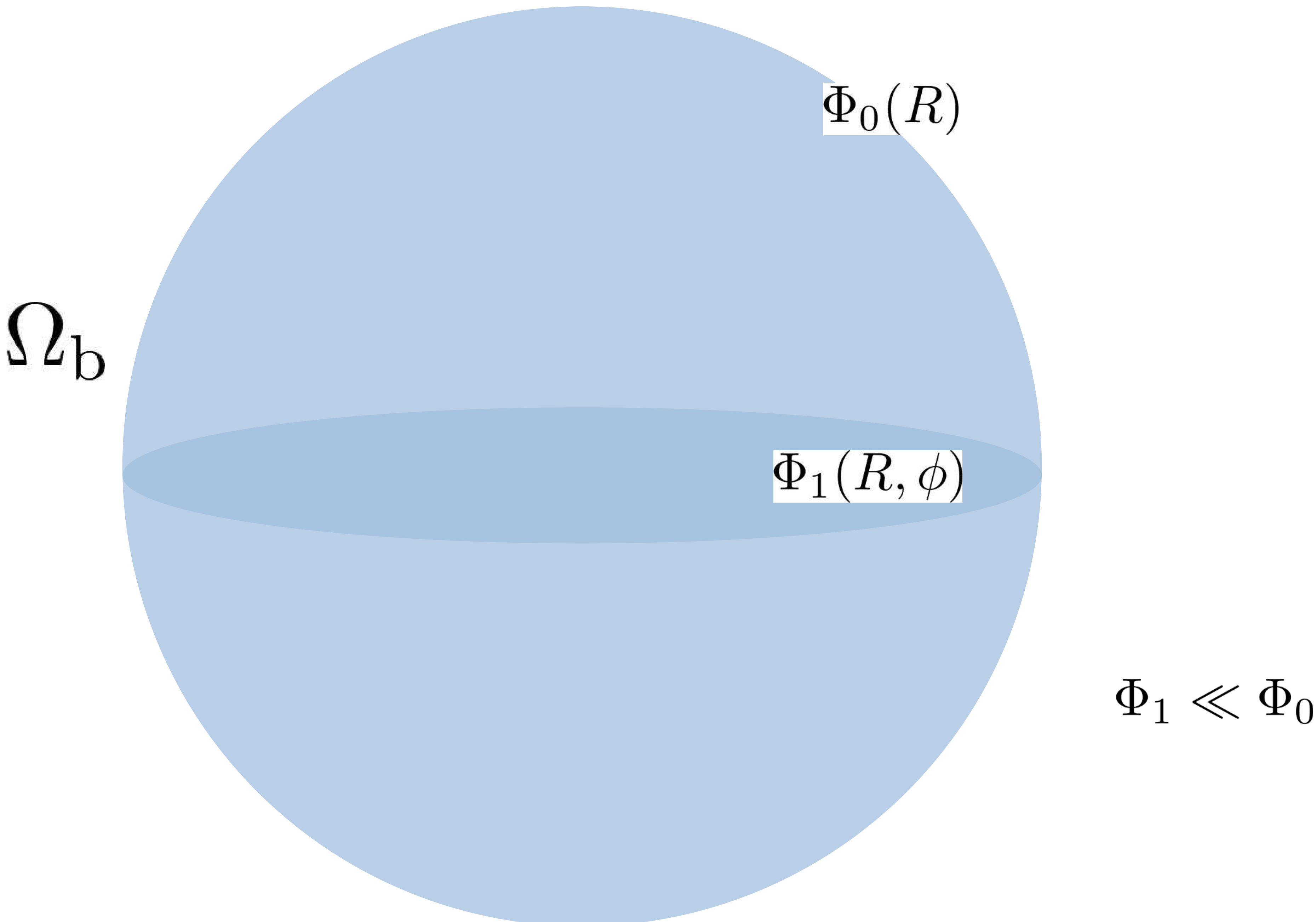
Assumptions

(1) A weak perturbation : $\phi(R,\varphi) = \underbrace{\phi_0(R)}_{\text{cyl. symmetry}} + \underbrace{\phi_1(R,\varphi)}_{\text{perturbation}}$ $\quad \dfrac{|\phi_1|}{|\phi_0|} \ll 1$

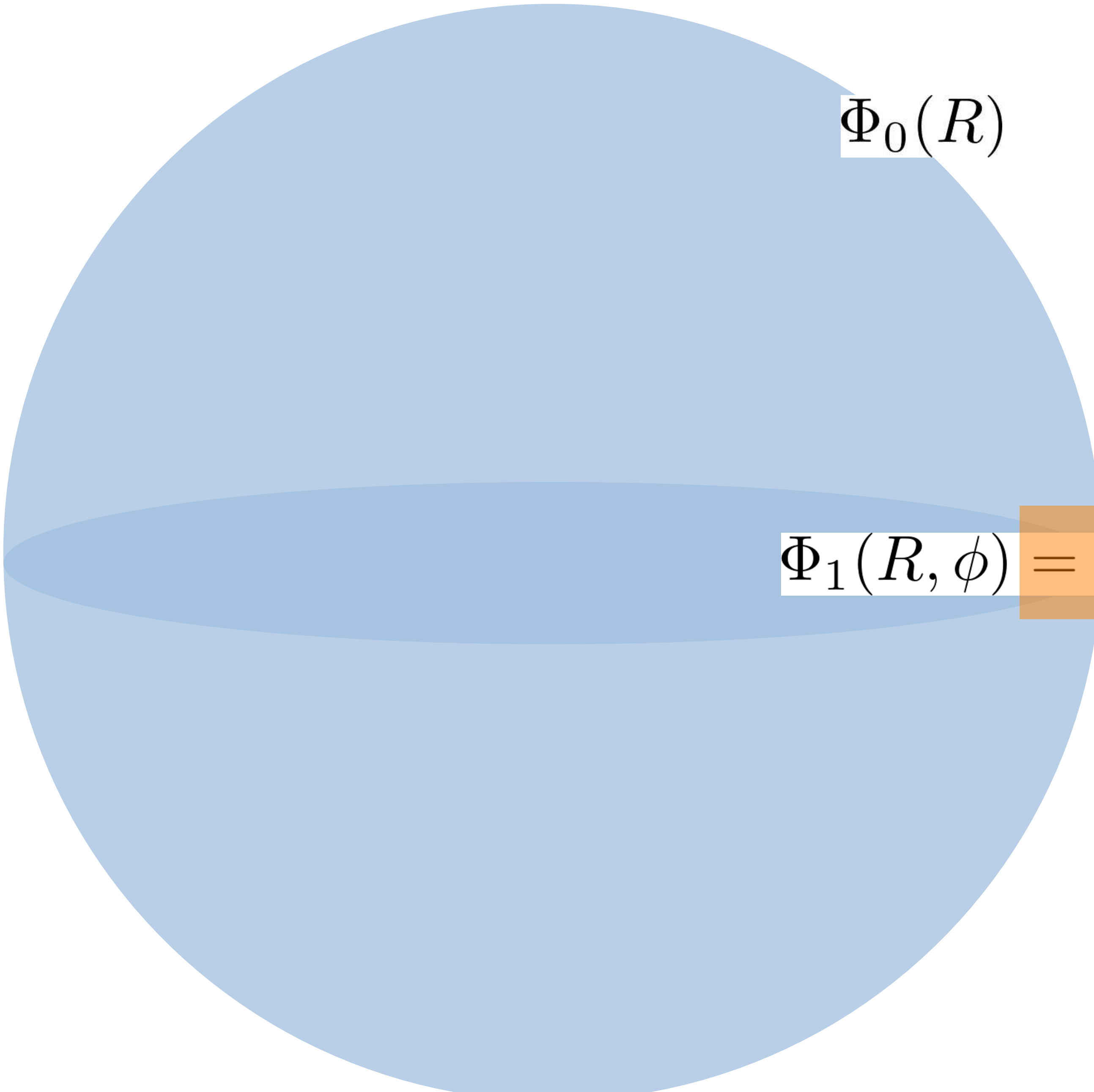
$$\phi_1(R,\varphi) = \underbrace{\phi_b(R)}_{\text{radial dependency}} \underbrace{\cos(m\varphi)}_{\text{azimuthal dependency}}$$

m : perturbation mode

The weakly-bared galaxy model



The weakly-bared galaxy model



Ω_{b}

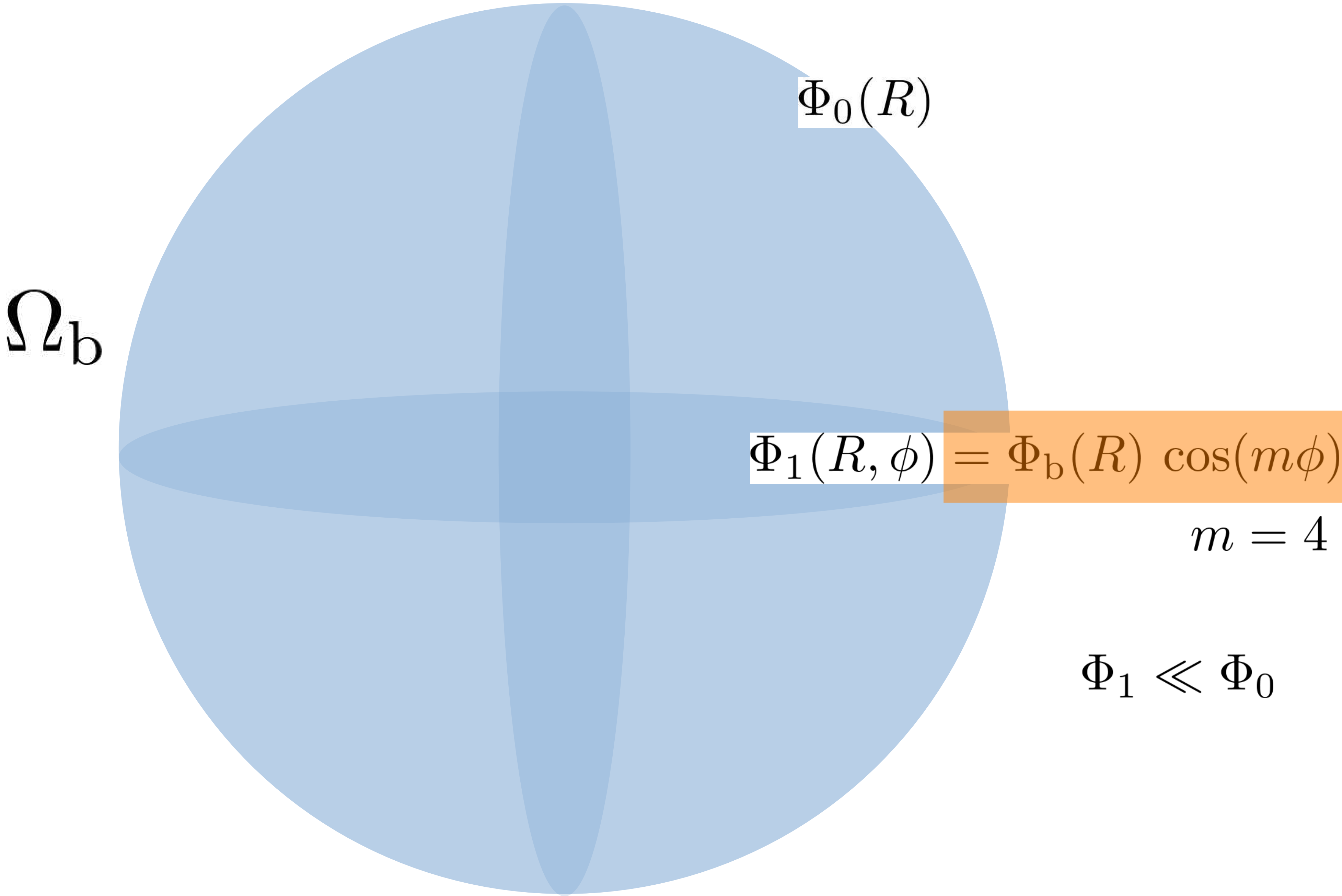
$\Phi_0(R)$

$$\Phi_1(R, \phi) = \Phi_{\mathrm{b}}(R)\, \cos(m\phi)$$

$m = 2$

$\Phi_1 \ll \Phi_0$

The weakly-bared galaxy model



Assumptions

② The motion may be decomposed into two parts

1) circular motion
2) perturbation

$$\begin{cases} R(t) = R_0(t) + R_1(t) \\ \varphi(t) = \varphi_0(t) + \varphi_1(t) \end{cases}$$

$R_1 \ll R_0$

$\varphi_1 \ll \varphi_0$

Note

$$\begin{cases} R_0(t) = R_0 \\ \varphi_0(t) = (\Omega_0 - \Omega_b)\, t \end{cases}$$

(R_0 = radius of the guiding center)

(Ω_0 = circular frequency)

Solution of the EoM (2nd order terms)

EXERCICE

Radial motion

$$R_1(\varphi_0) = C_1 \cos\left(\frac{\kappa_0\, \varphi_0}{\Omega_0 - \Omega_b} + \alpha\right) - \left[\frac{d\Phi_b}{dR} + \frac{2\,\Omega\, \Phi_b}{R(\Omega - \Omega_b)}\right]_{R_0} \frac{\cos(m\, \varphi_0)}{\kappa_0^2 - m^2(\Omega_0 - \Omega_b)^2}$$

C_1, α : arbitrary constants

κ_0 : radial epicycle frequency

Azimuthal motion

$$\varphi_1(t) = -2\Omega_0 \frac{R_1}{R_0} - \frac{\Phi_b(R_0)}{R_0^2\,(\Omega_0 - \Omega_b)} \cos\left(m(\Omega_0 - \Omega_b)t\right) + cte$$

Discussion

$$R_1(\varphi_0) = C_1 \cos\left(\frac{\varkappa_0 \varphi_0}{\Omega_0 - \Omega_b} + \alpha\right) - \left[\frac{d\Phi_b}{dR} + \frac{2\Omega \Phi_b}{R(\Omega - \Omega_b)}\right]_{R_0} \frac{\cos(m\varphi_0)}{\varkappa_0^2 - m^2(\Omega_0 - \Omega_b)^2}$$

① if $\Phi_b(R) = 0$ (no perturbation)

$\varphi_0 = (\Omega_0 - \Omega_b)\,t$

Epicycles motions

$$R_1(t) = C_1 \cos(\varkappa_0 t + \alpha) \quad \equiv x(t) \text{ radial oscillations}$$

$$\dot{\varphi}_1(t) = -2\Omega_0 \frac{R_1(t)}{R_0} \quad \Rightarrow y(t) \text{ oscillations along the orbit}$$

② if $C_1 = 0$, $\Phi_b \neq 0$

$$R_1(\varphi_0) = -\left[\frac{d\Phi_b}{dR} + \frac{2\Omega \Phi_b}{R(\Omega - \Omega_b)}\right]_{R_0} \frac{\cos(m\varphi_0)}{\varkappa_0^2 - m^2(\Omega_0 - \Omega_b)^2}$$

cte — periodic in φ_0 $\left(\frac{2\pi}{m}\right)$

$\Rightarrow$ closed orbit

③ if $C_1 \neq 0$ oscillations around the closed orbit (same family)

The orbit is not necessary closed

Resonances

! two problematic terms $\frac{1}{\Omega_0 - \Omega_b}$ and $\frac{1}{\varkappa_0^2 - m^2(\Omega_0 - \Omega_b)^2}$

$\Rightarrow$ R_1 may diverge !

1) $\boxed{\Omega_0 = \Omega_b}$ Corotation

we are at a radius where the circular frequency is similar to the pattern speed of the bar

as $\dot{\varphi}_0 = \Omega_0 - \Omega_b \Rightarrow \dot{\varphi}_0 = 0$ → static in the rotating frame

2) $\boxed{m(\Omega_0 - \Omega_b) = \pm \varkappa_0}$ Lindblad resonances

$m(\Omega_0 - \Omega_b)$: frequ. at which the star encounter the potential minimum

→ the frequency at wich a star encounter a potential minimum is similar to its radial frequency

$\Rightarrow$ exitation

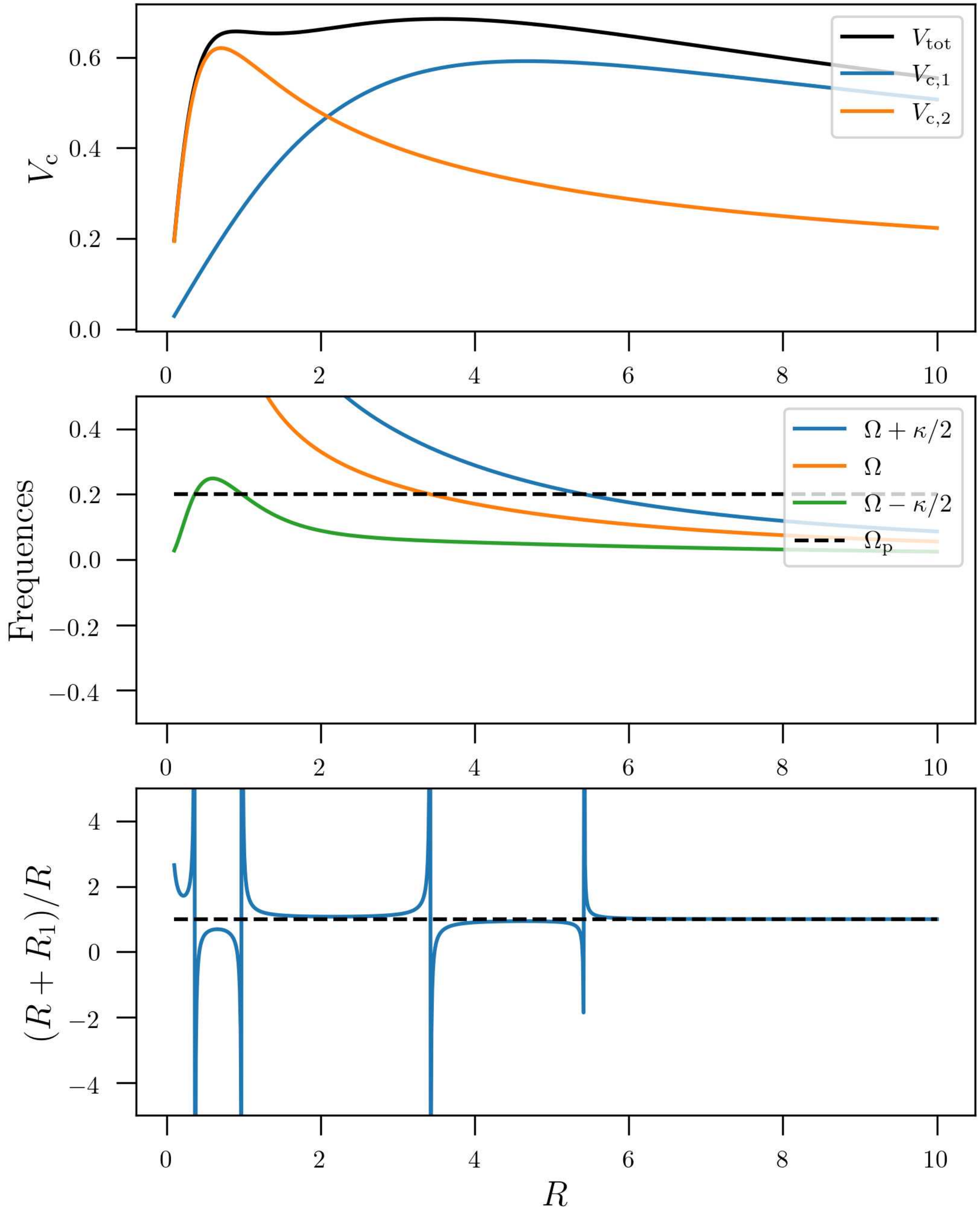
$\equiv \Omega_b = \Omega \pm \frac{\varkappa}{2}$

A circular orbit has two natural frequencies

① $\varkappa$: radial frequ.

② 0 : azimuthal frequ. (no change $\Rightarrow$ frequ $= 0$)

Resonances occur when the forcing frequency $m(\Omega_0 - \Omega_b)$ is equal to one of these frequencies.



Disk : Miyamoto-Nagai
Bulge : Plummer

Inner Lindblad resonnances
(ILR1, ILR2)

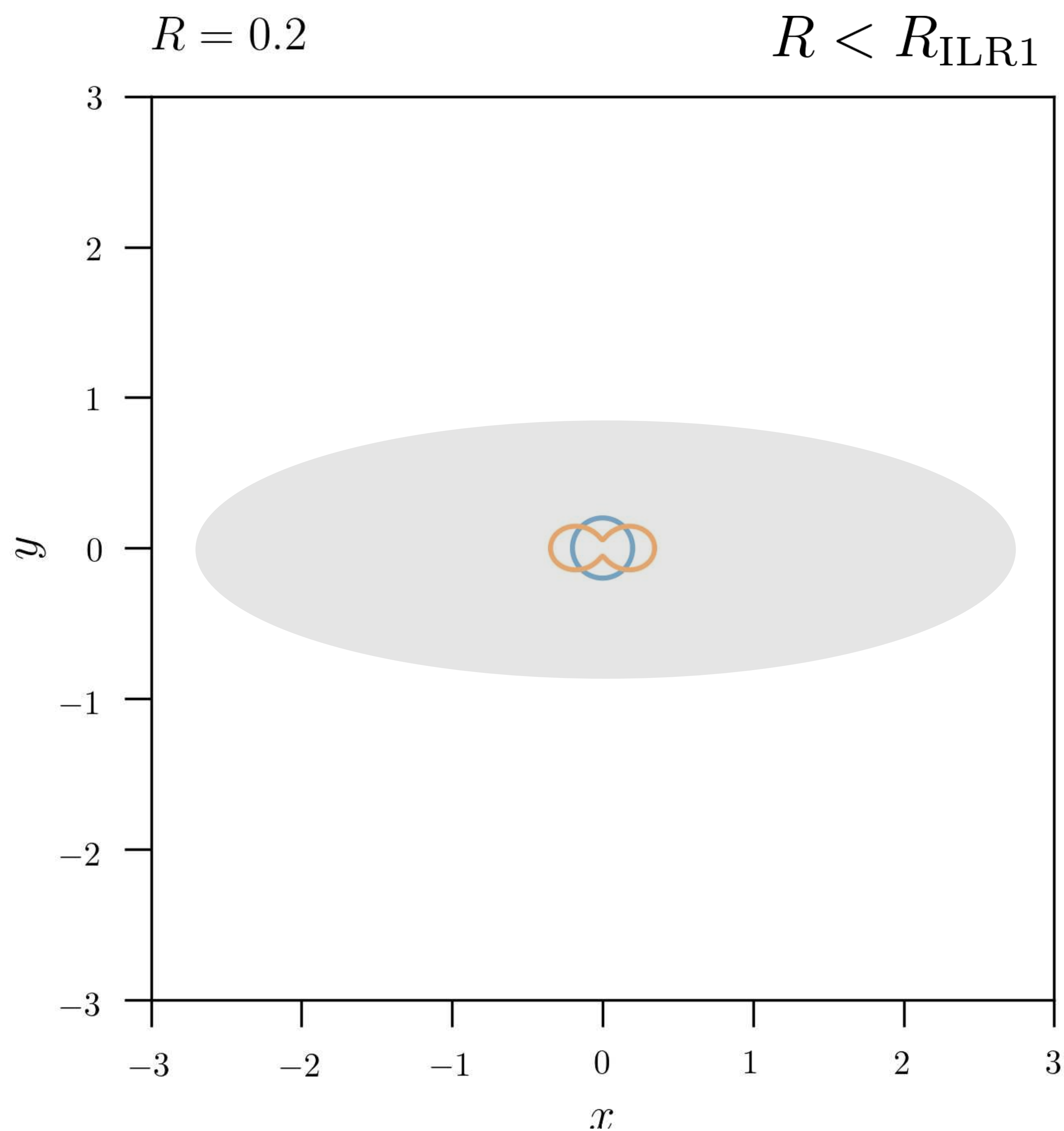
$$\Omega_{\rm b} = \Omega - \kappa/2$$

Outer Lindblad resonnance
(OLR)

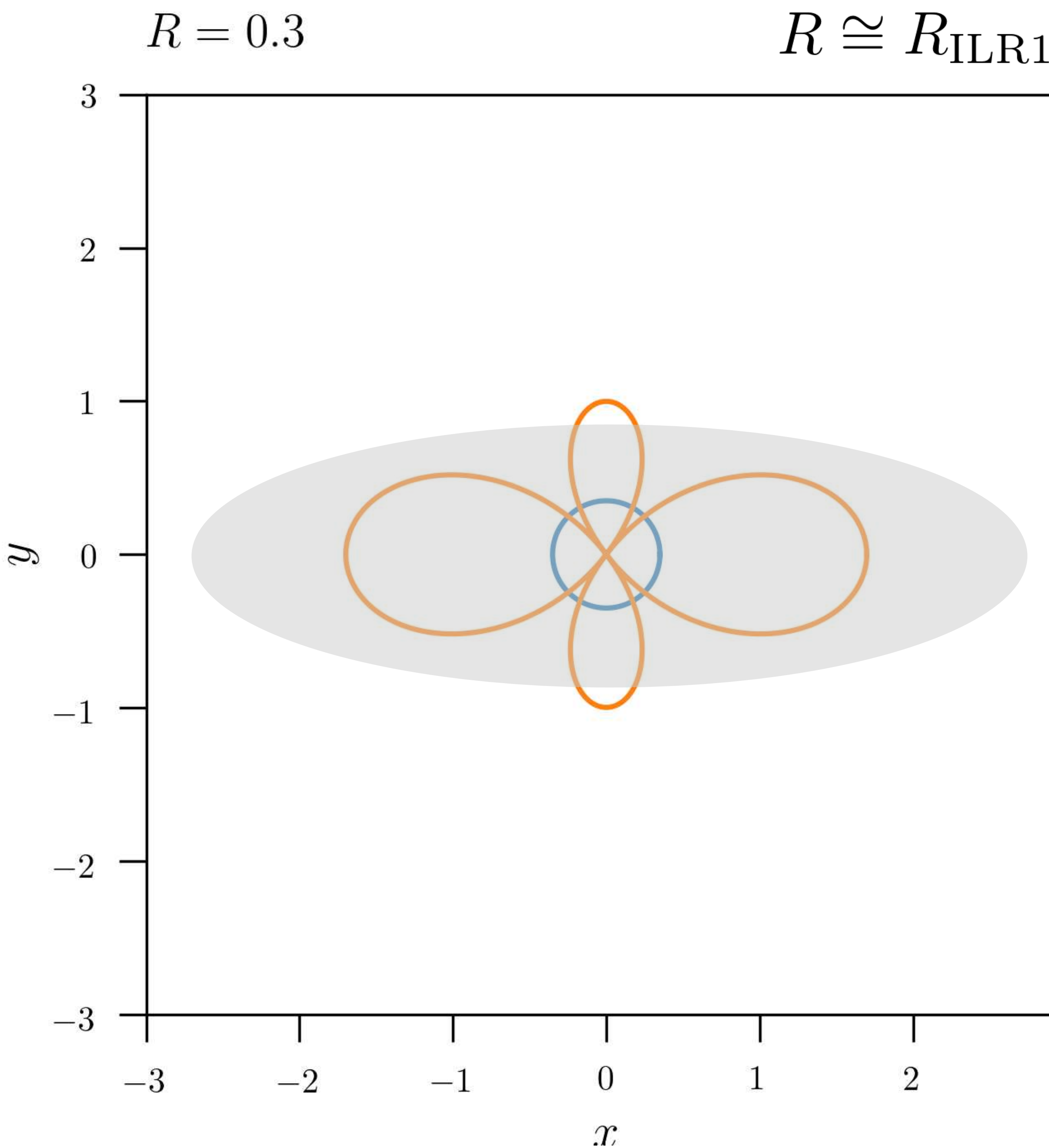
$$\Omega_{\rm b} = \Omega + \kappa/2$$

Corotation (CR)

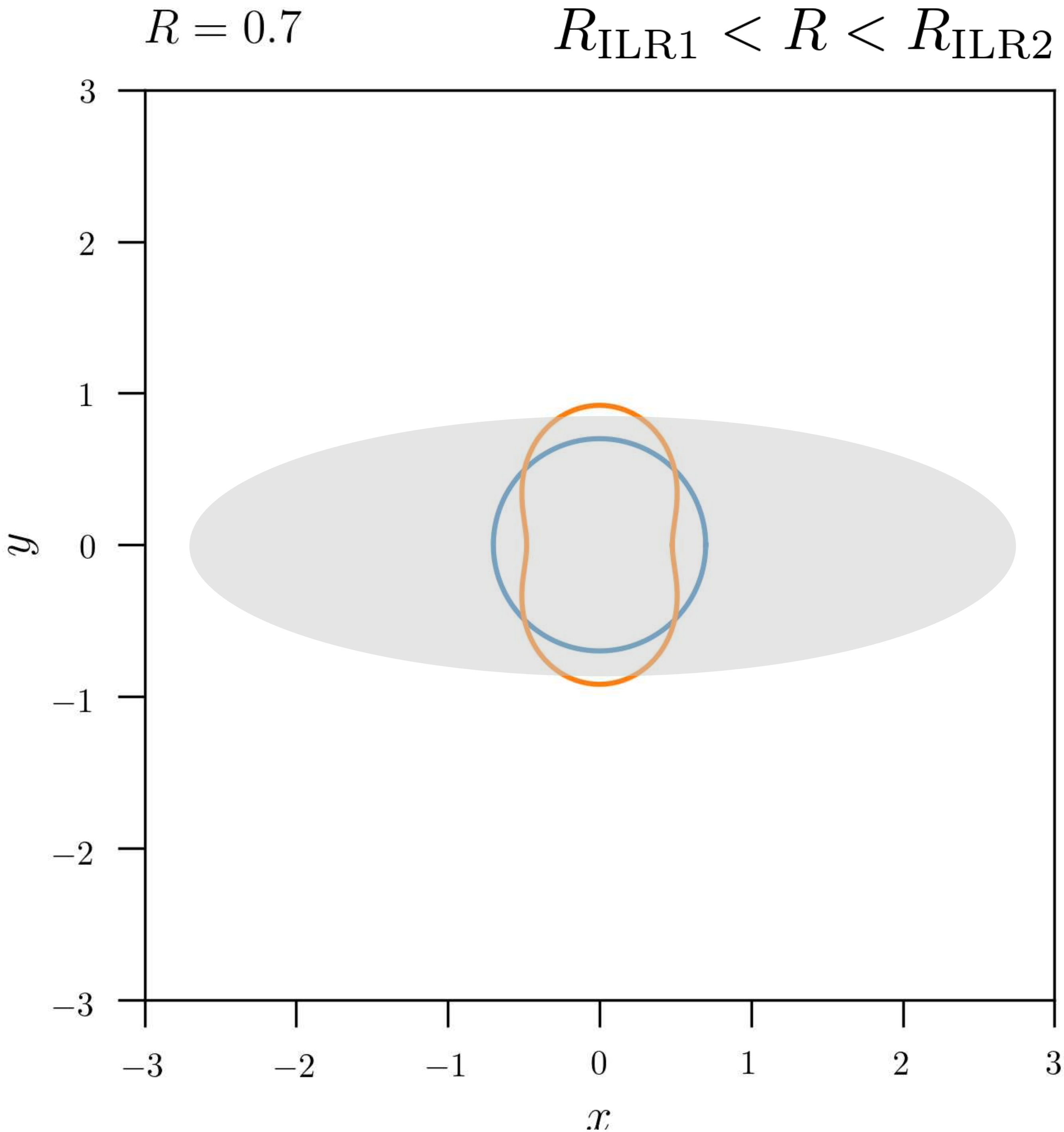
$$\Omega_{\rm b} = \Omega$$

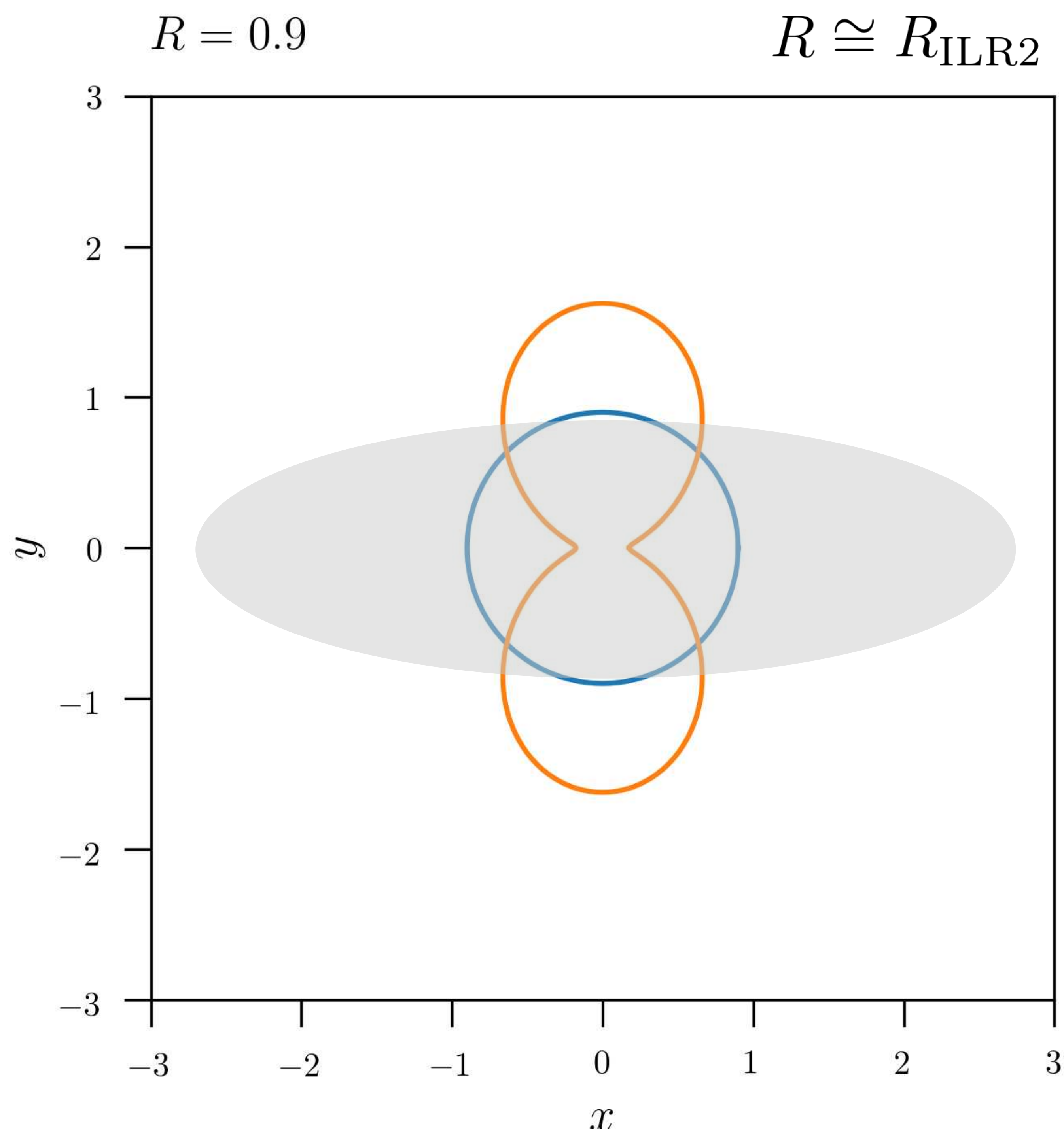


$R = 0.2$
$R < R_{\mathrm{ILR1}}$
3
2
1
0
−1
−2
−3
y
−3
−2
−1
0
1
2
3
x

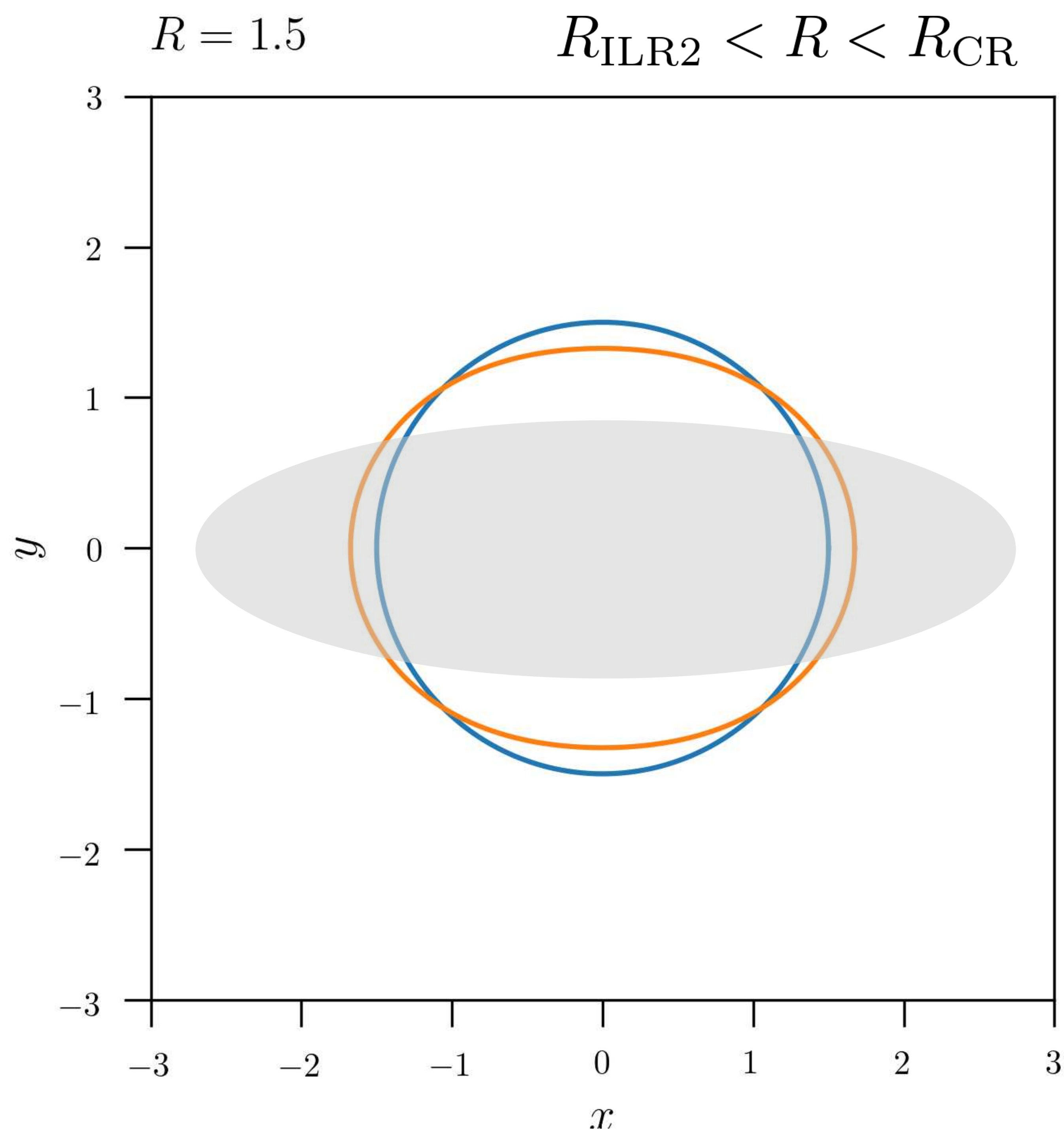


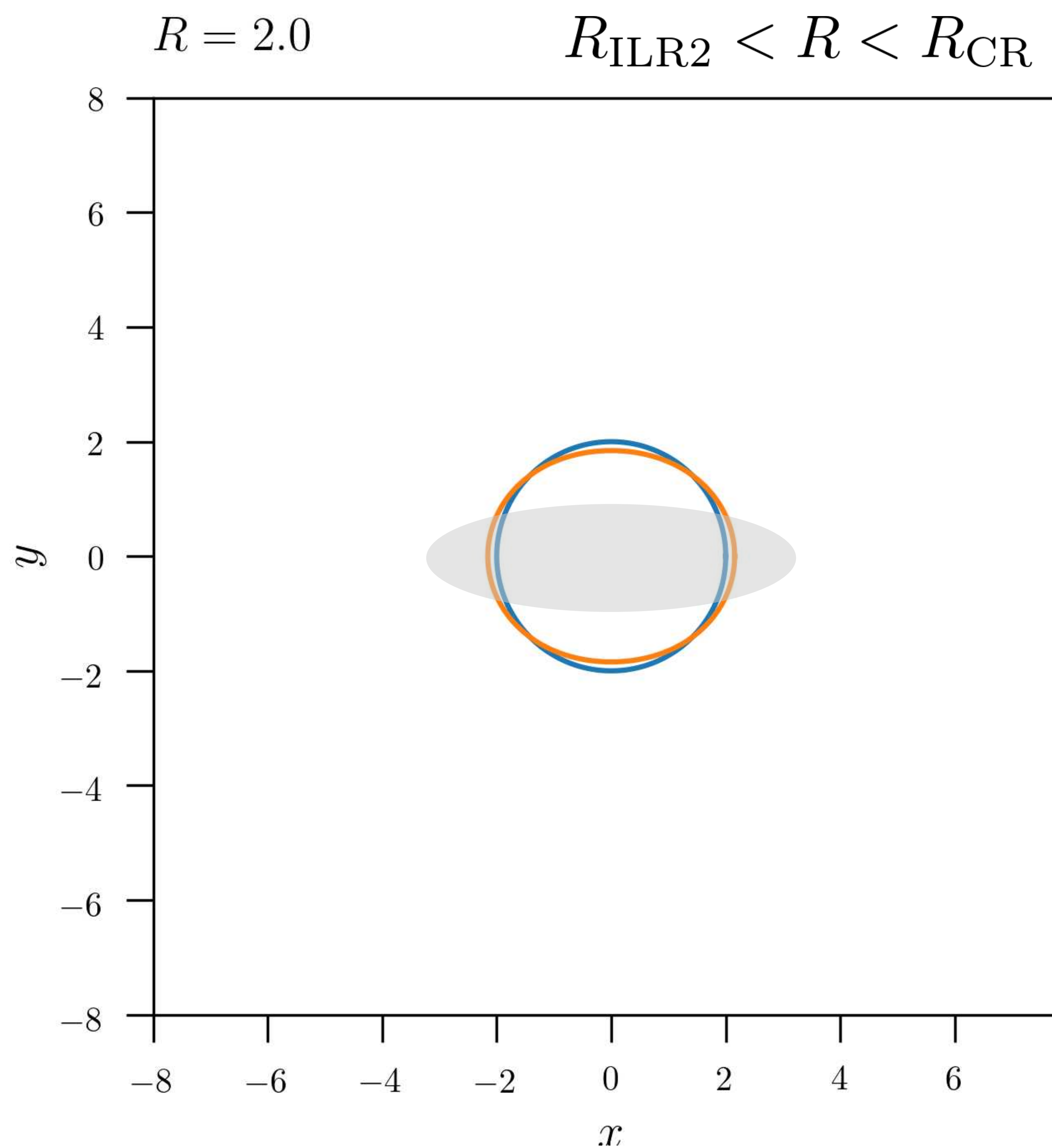
$R = 0.3$
$R \cong R_{\mathrm{ILR1}}$
3
2
1
0
−1
−2
−3
y
−3
−2
−1
0
1
2
3
x

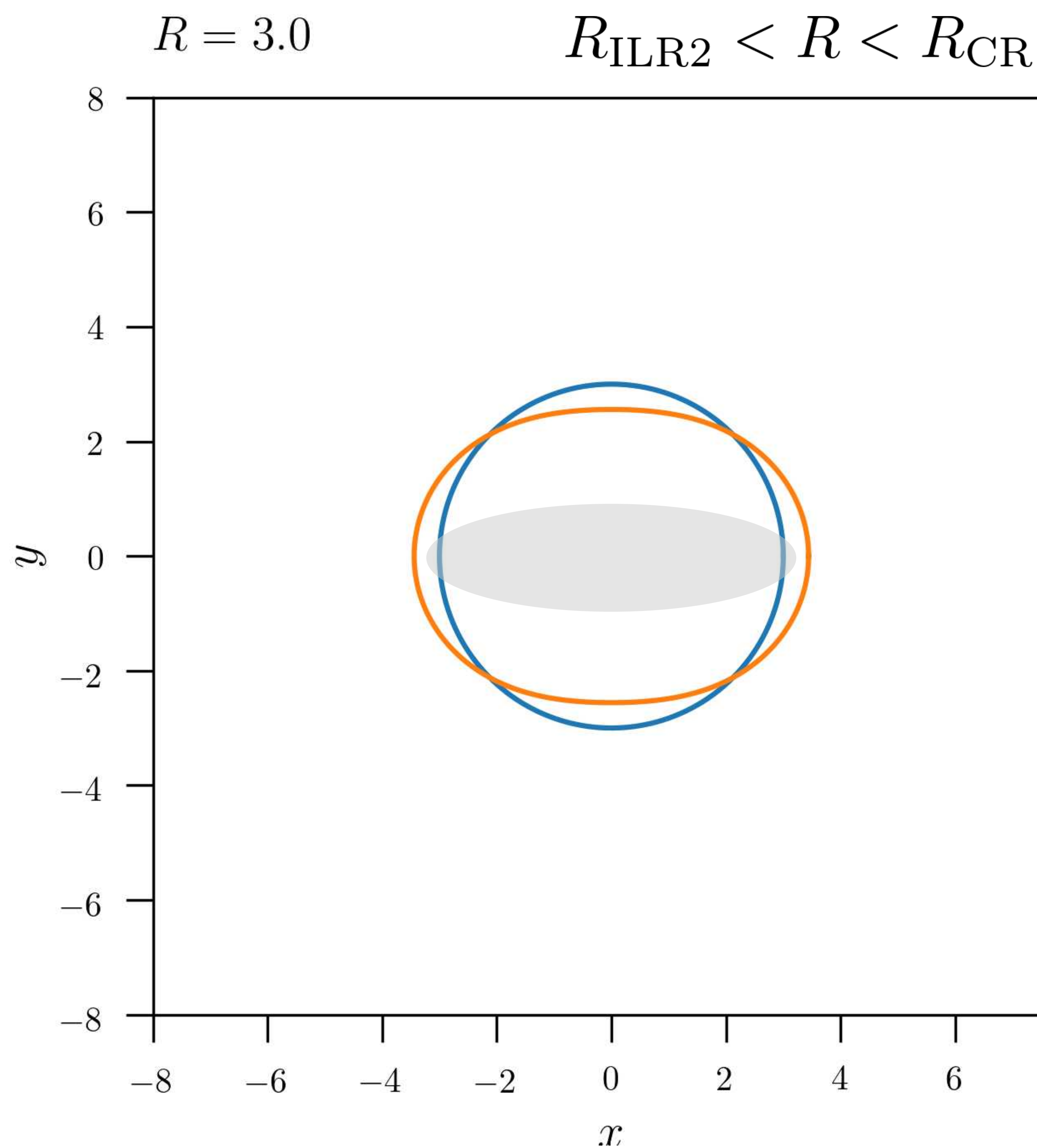

$R = 0.7$
$R_{\mathrm{ILR1}} < R < R_{\mathrm{ILR2}}$
3
2
1
0
−1
−2
−3
y
x



$R = 0.9$
$R \cong R_{\mathrm{ILR2}}$
3
2
1
0
−1
−2
−3
y
−3
−2
−1
0
1
2
3
x

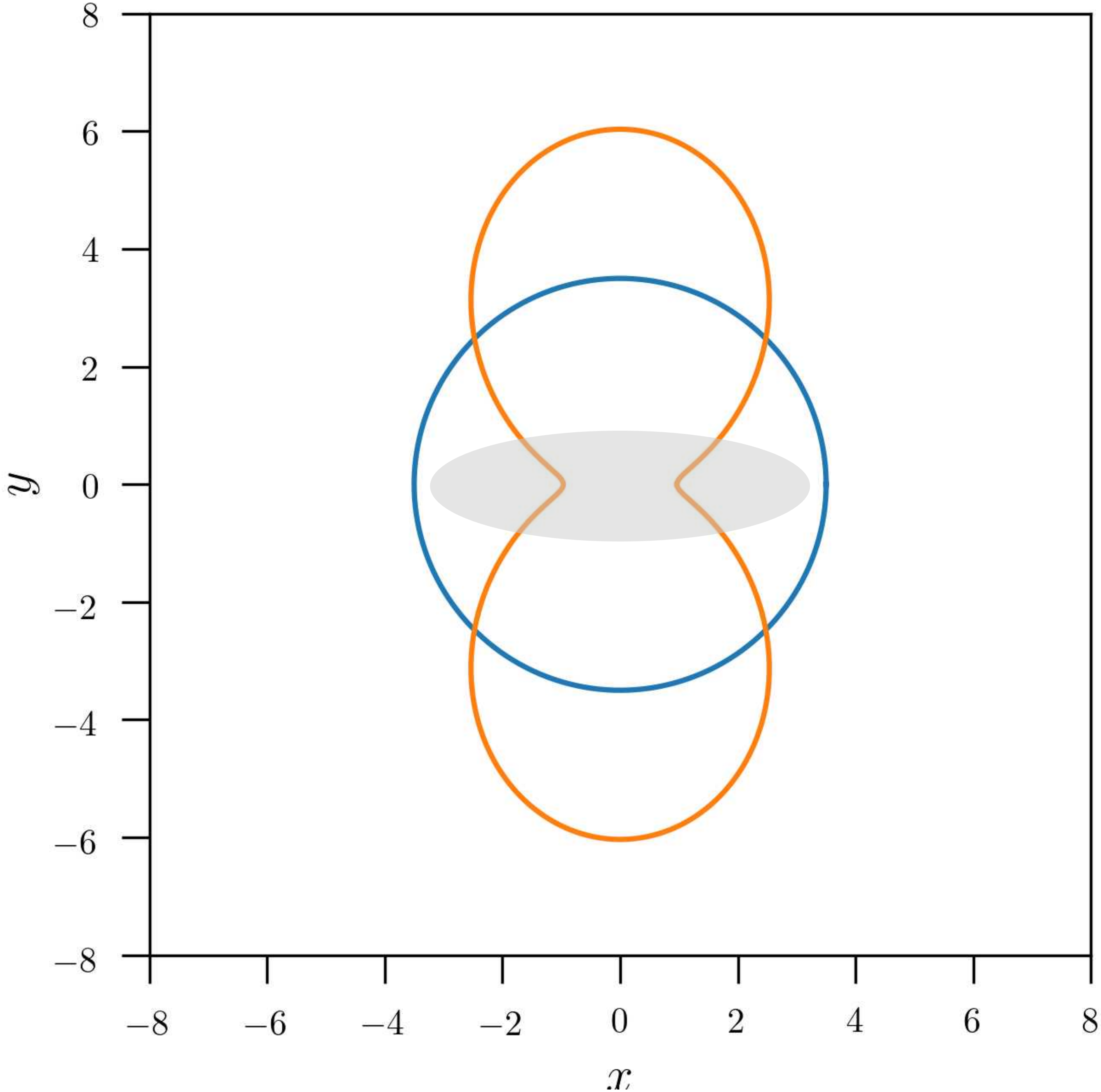

$R = 1.5$
$R_{\mathrm{ILR2}} < R < R_{\mathrm{CR}}$
3
2
1
0
−1
−2
−3
y
−3
−2
−1
0
1
2
3
x


$R = 2.0$
$R_{\mathrm{ILR2}} < R < R_{\mathrm{CR}}$
8
6
4
2
0
−2
−4
−6
−8
y
x

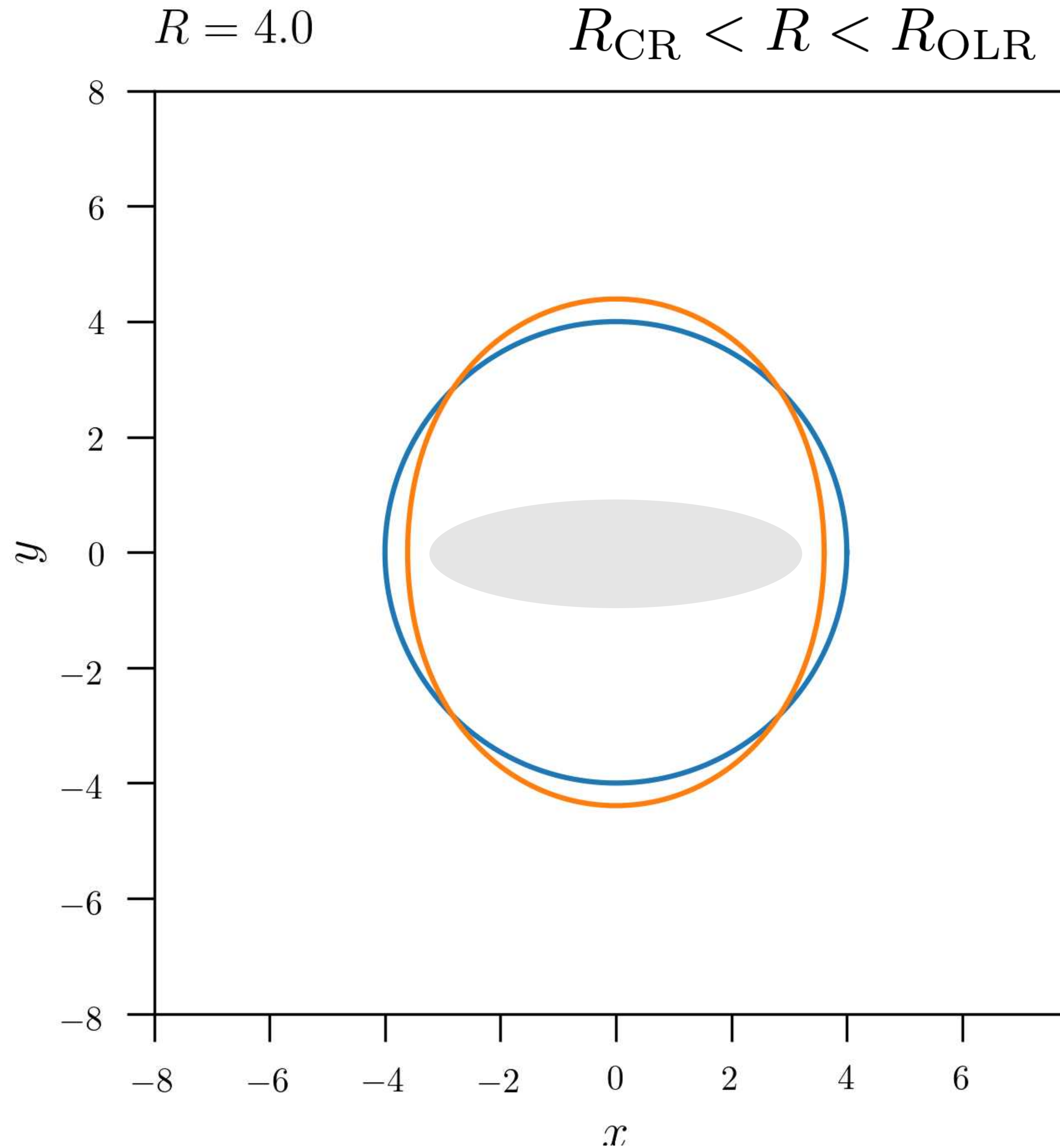

$R = 3.0$
$R_{\rm ILR2} < R < R_{\rm CR}$
8
6
4
2
0
−2
−4
−6
−8
y
−8
−6
−4
−2
0
2
4
6
8
x

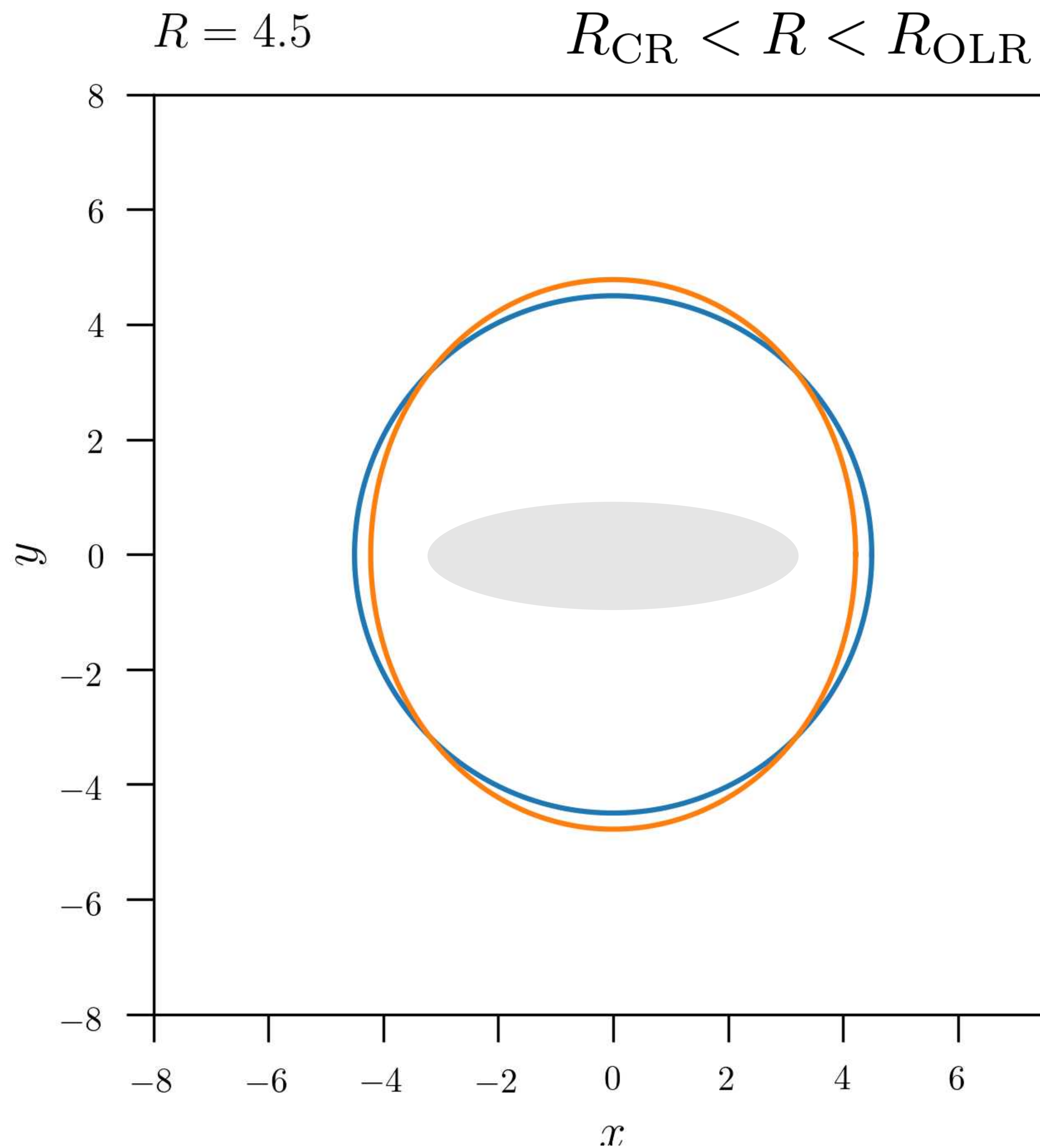
$R = 3.5$

$R \cong R_{\mathrm{CR}}$

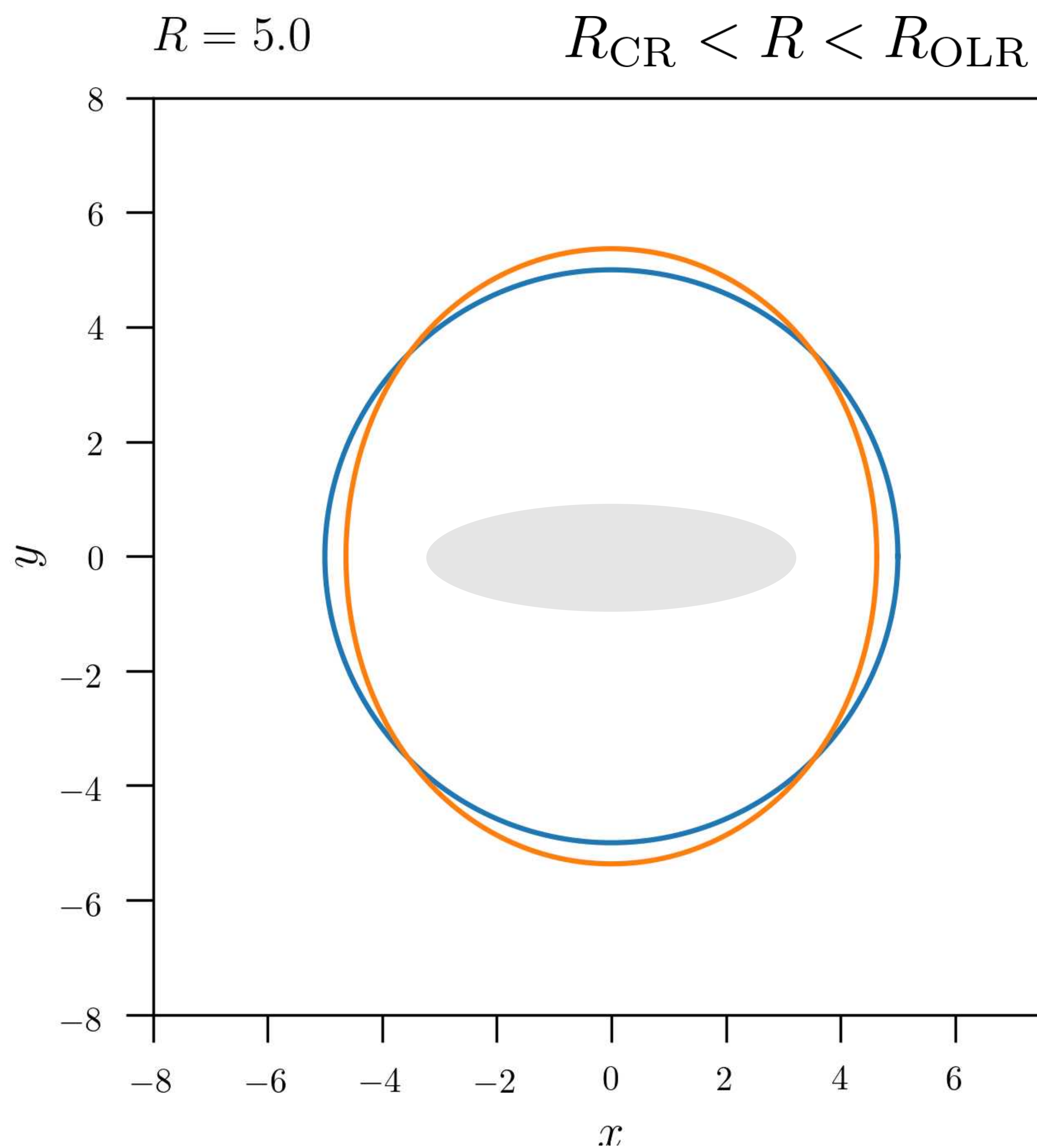


+ closed orbits around L4, L5

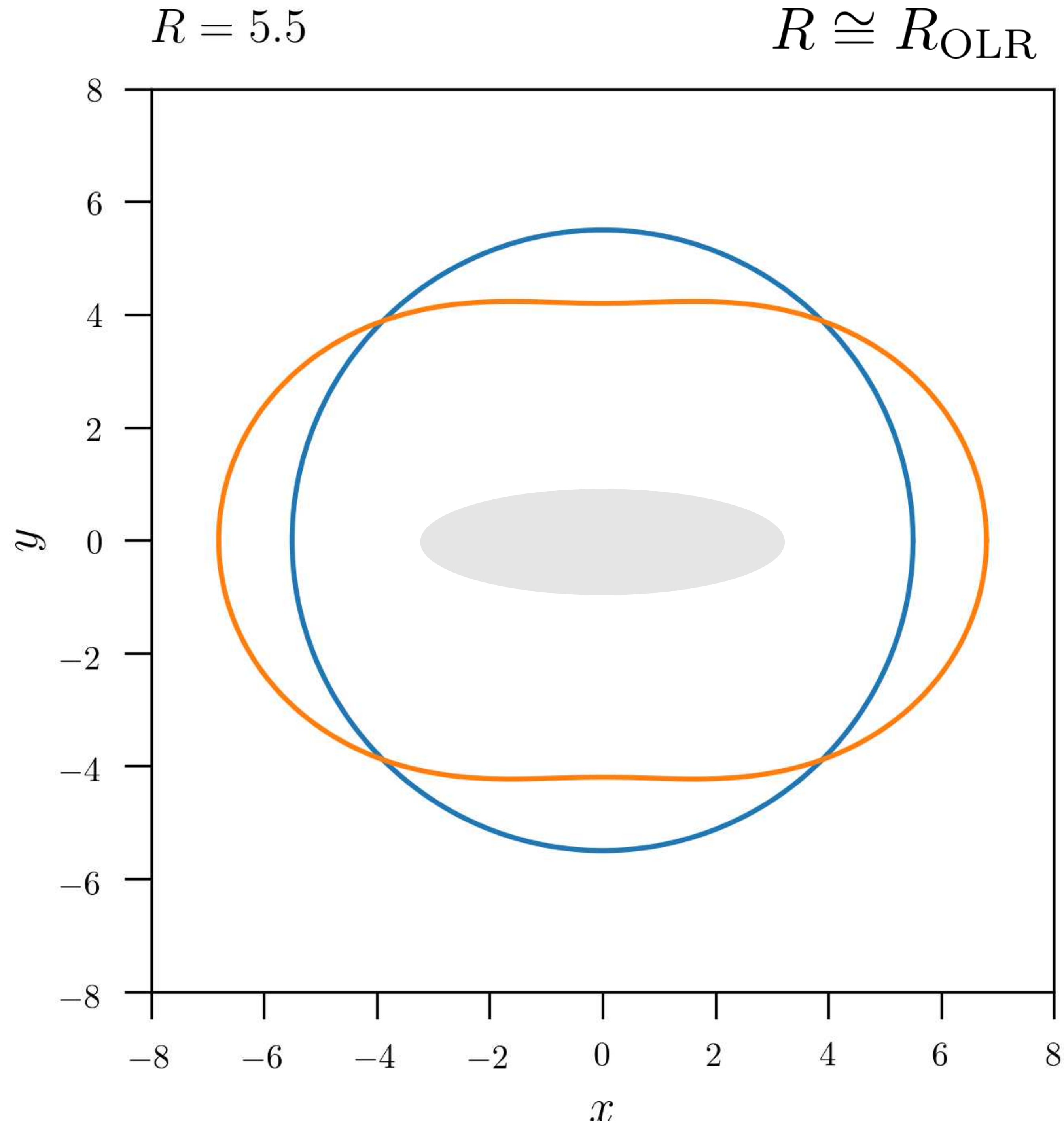

$R = 4.0$
$R_{\mathrm{CR}} < R < R_{\mathrm{OLR}}$
8
6
4
2
0
−2
−4
−6
−8
y
−8
−6
−4
−2
0
2
4
6
8
x



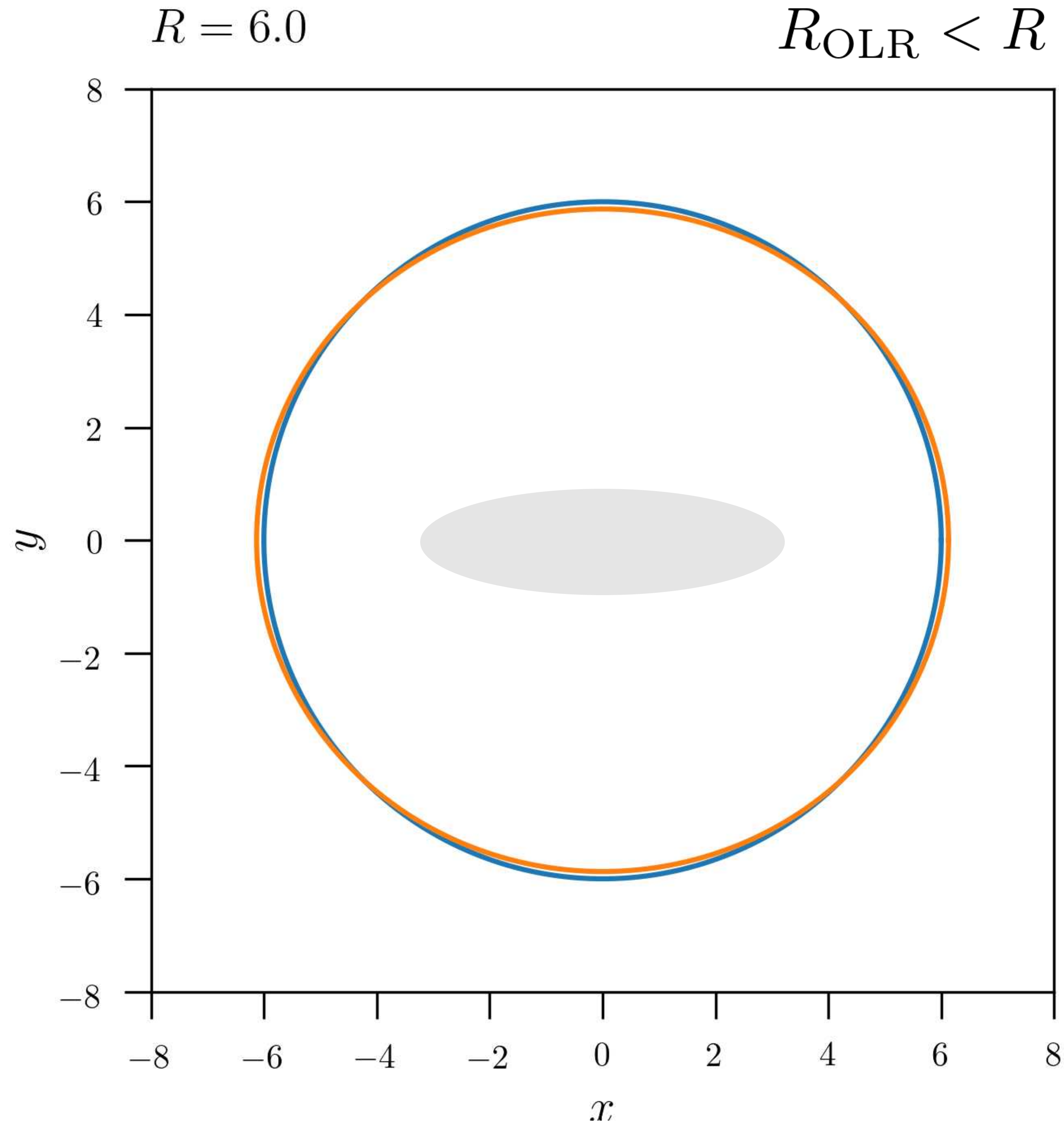
$R = 4.5$
$R_{\mathrm{CR}} < R < R_{\mathrm{OLR}}$
8
6
4
2
0
−2
−4
−6
−8
y
x



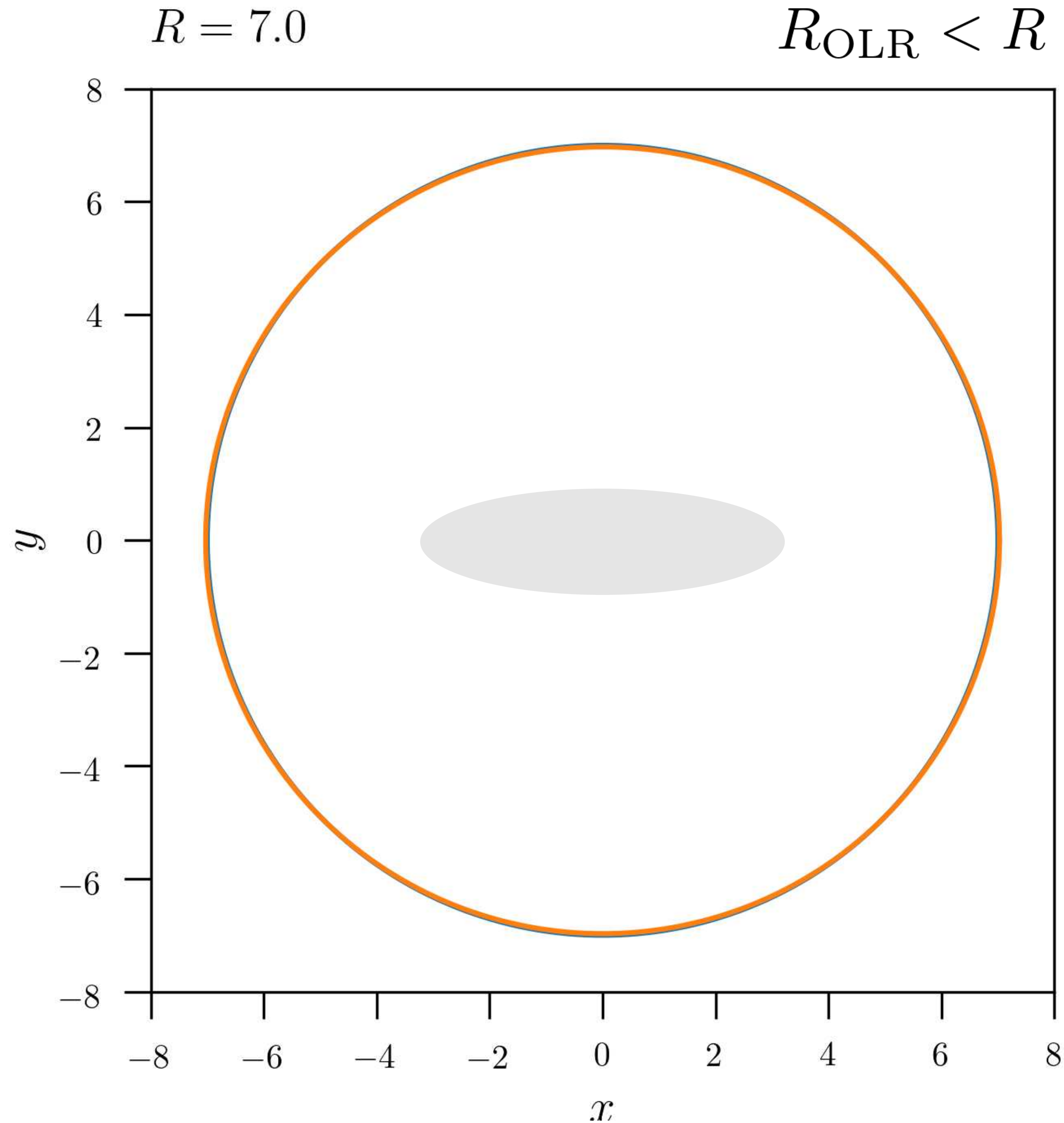
$R = 5.0$
$R_{\mathrm{CR}} < R < R_{\mathrm{OLR}}$
8
6
4
2
0
−2
−4
−6
−8
y
−8
−6
−4
−2
0
2
4
6
8
x



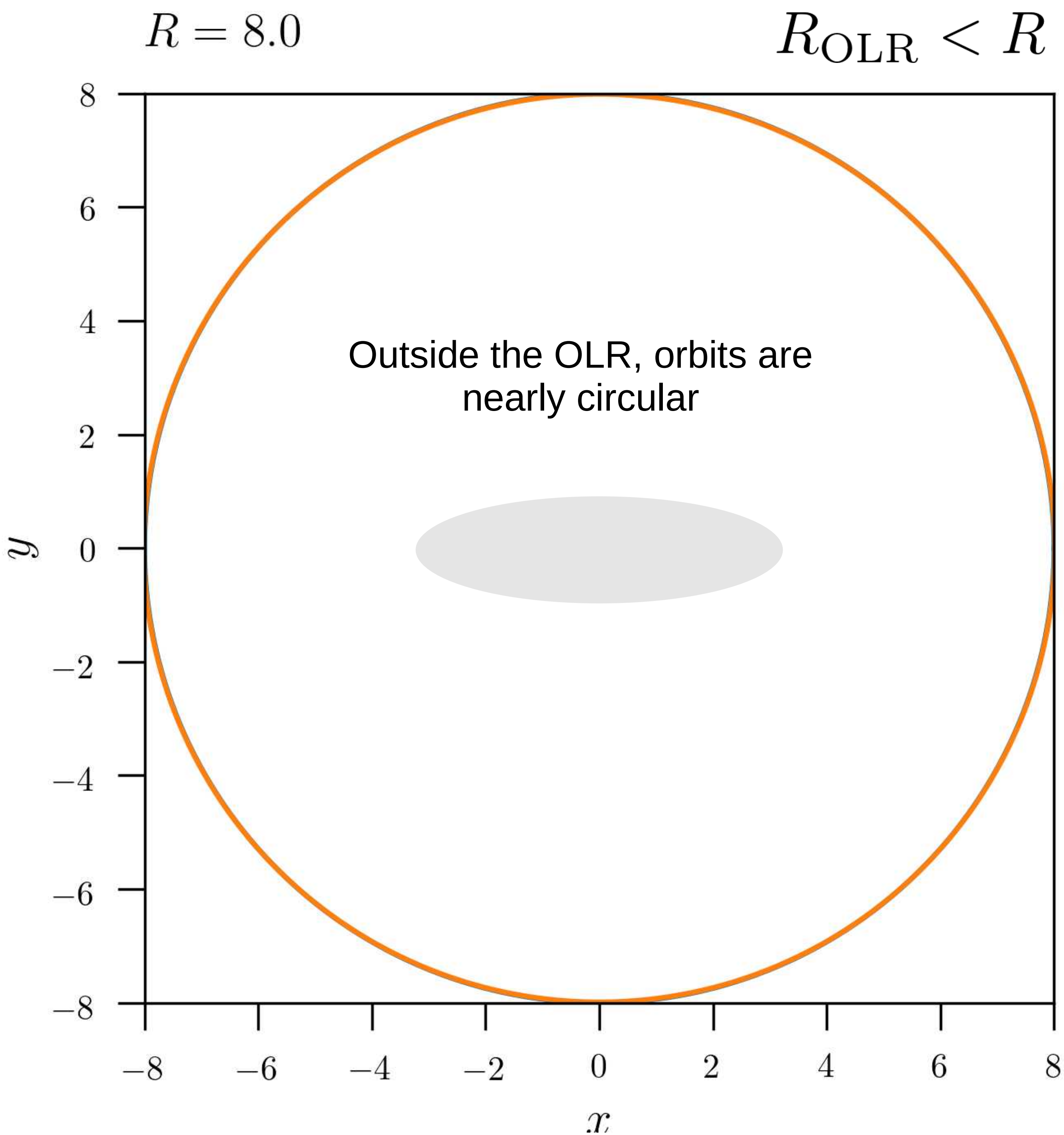
$R = 5.5$
$R \cong R_{\mathrm{OLR}}$
8
6
4
2
0
−2
−4
−6
−8
y
−8
−6
−4
−2
0
2
4
6
8
x



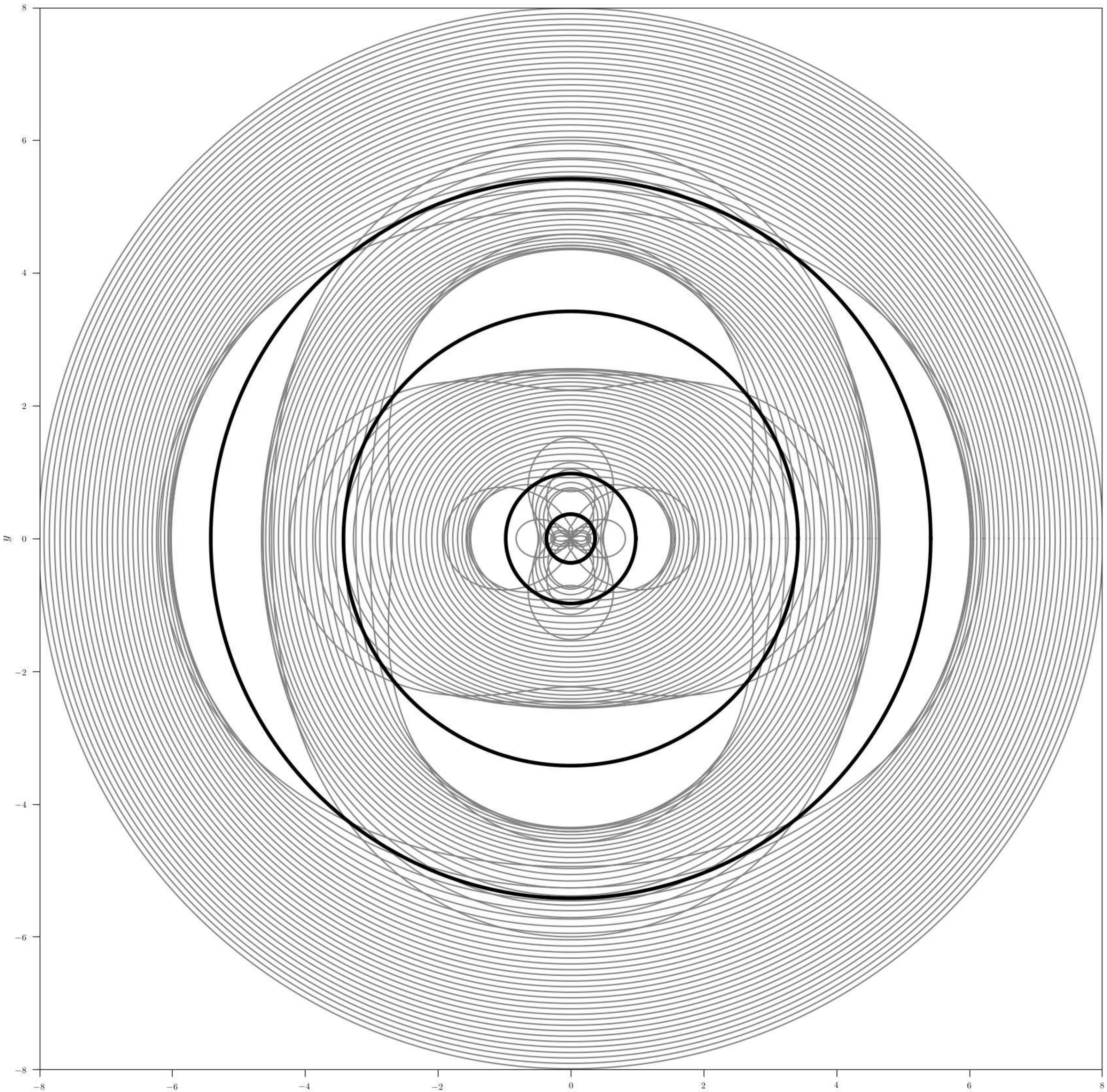
$R = 6.0$
$R_{\mathrm{OLR}} < R$
y
x
8
6
4
2
0
−2
−4
−6
−8



$R = 7.0$
$R_{\rm OLR} < R$
8
6
4
2
0
−2
−4
−6
−8
y
−8
−6
−4
−2
0
2
4
6
8
x

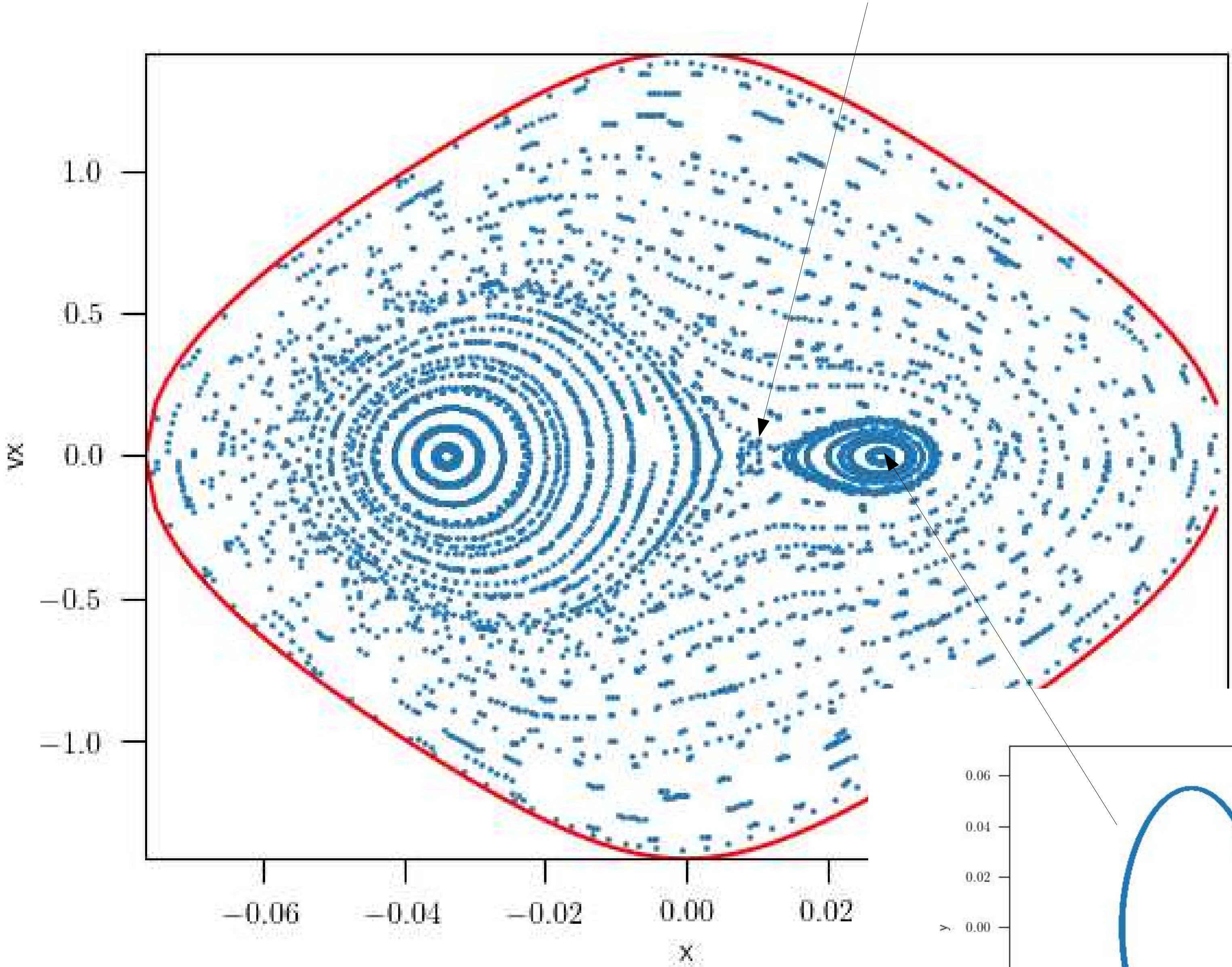


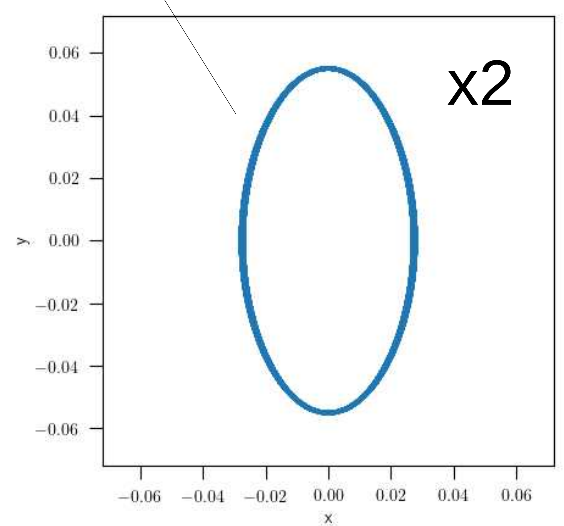
$R = 8.0$
$R_{\mathrm{OLR}} < R$
Outside the OLR, orbits are nearly circular
8
6
4
2
0
−2
−4
−6
−8
−8
−6
−4
−2
0
2
4
6
8
x
y



8
6
4
2
0
−2
−4
−6
−8
y
−8
−6
−4
−2
0
2
4
6
8

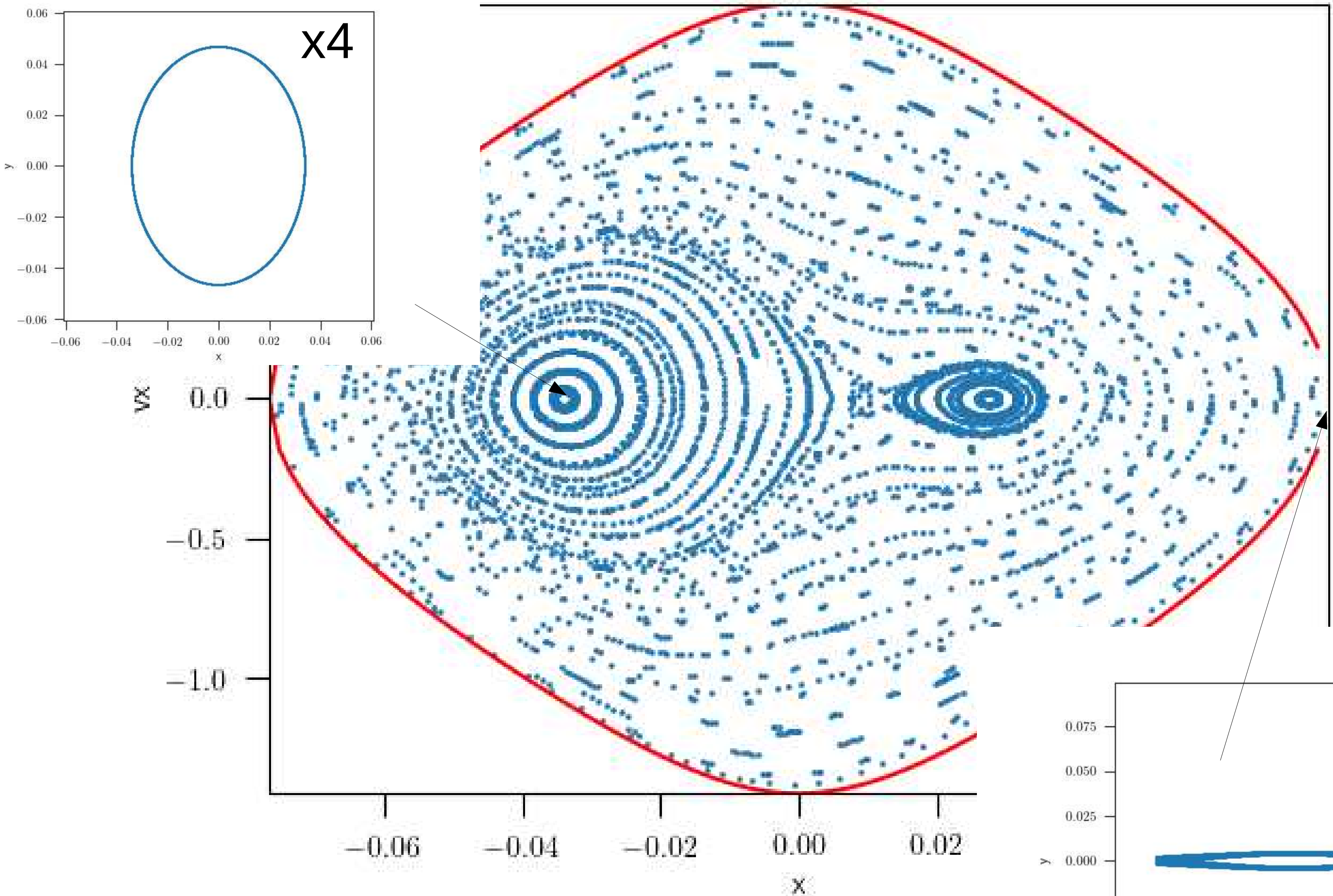
Bifurcation : apparition of x2 (stabe)/x3 (unstable) orbits

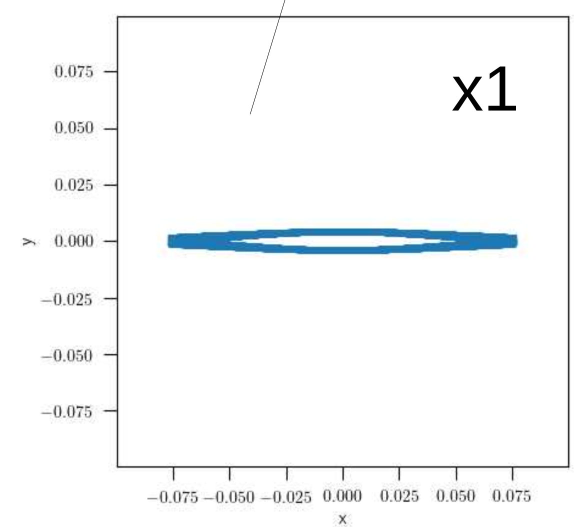





```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0268
```

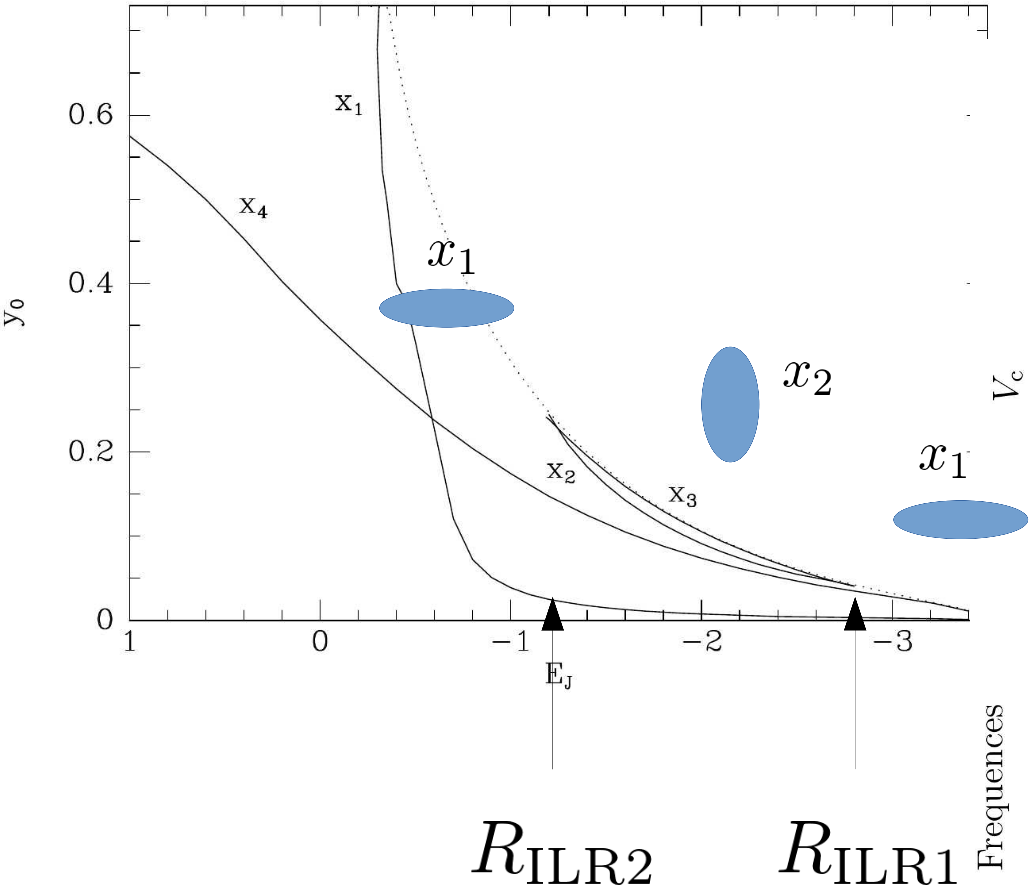
x1 : prograde x4 : retrograde

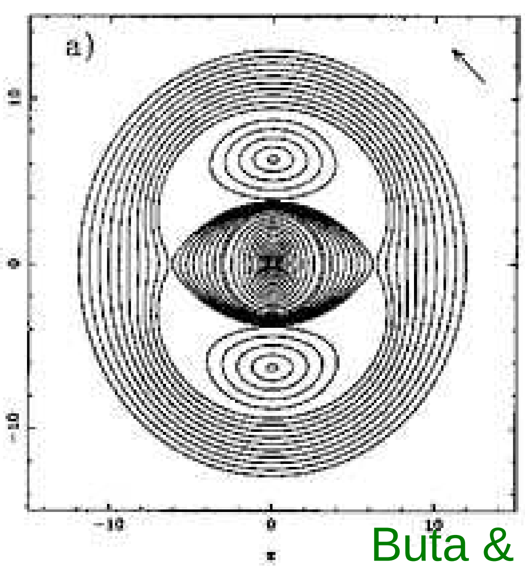




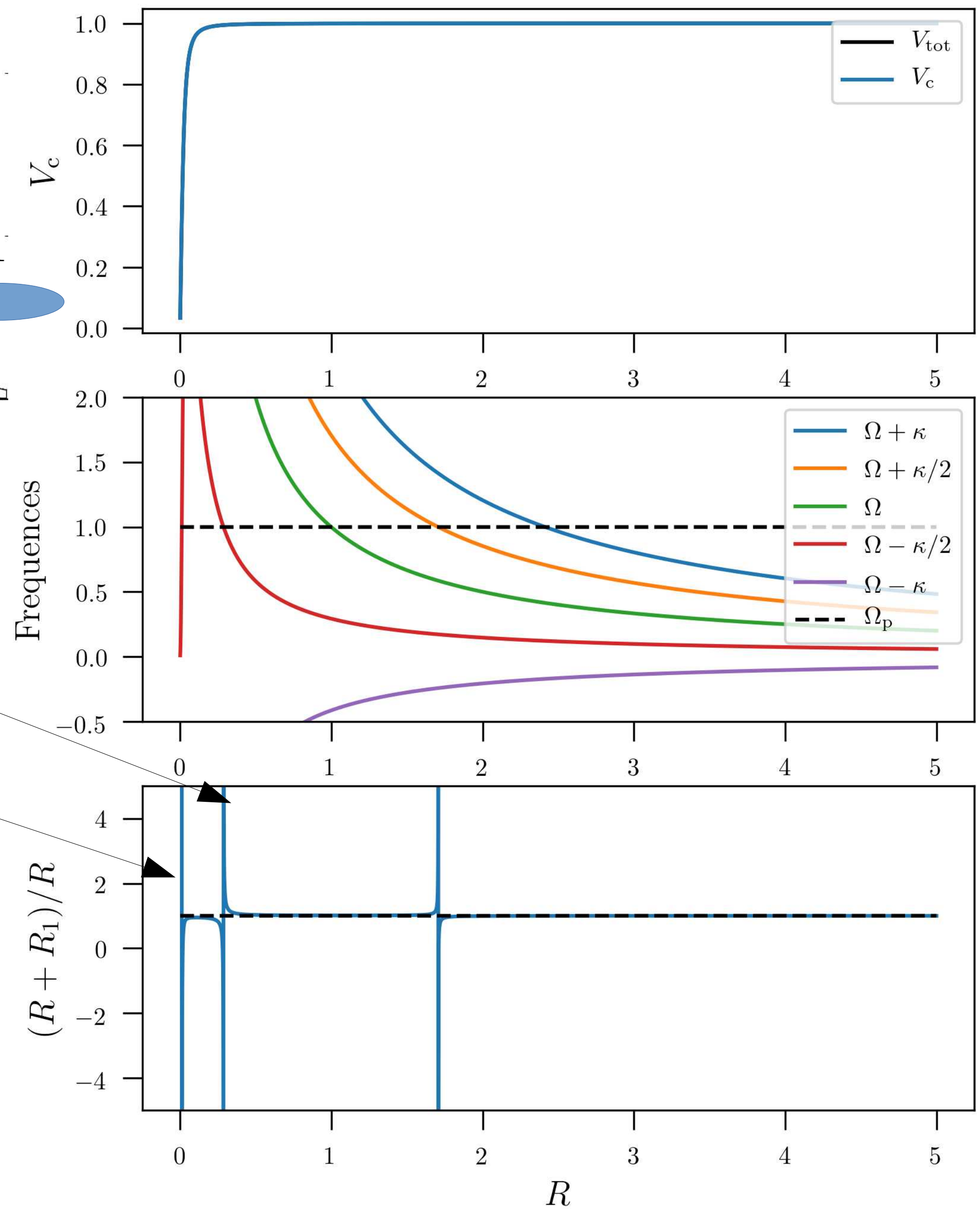

```
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x 0.0766659
./mapping.py --V0 1. --q 0.8 --Rc 0.03 --Omega 1 -E -2.5  --x -0.034
```

Lindblad frequencies for the Logarithmic potential





Buta & Combes 1998



The End