Astrophysics IV, Dr. Yves Revaz — EPFL

4th year physics — Exercises week 9

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Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

In two dimensions, assuming a bar rotating with a pattern speed Ω_b, in cylindrical coordinates, the Lagrangian writes: 2D, with $\vec{\Omega}_b = \Omega_b \vec{e}_z$ gives

$$L\left(R, \dot{R}, \theta, \dot{\theta}\right) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\left(R\left(\dot{\theta} + \Omega_b\right)\right)^2 - \phi\left(R, \theta\right). \tag{1}$$

The equation of motion is derived using the Euler-Lagrange equation:

$$\begin{cases} \ddot{R} = R\left(\dot{\theta} + \Omega_b\right)^2 - \frac{\partial \phi}{\partial R} \\ \frac{d}{dt}\left(R^2\left(\dot{\theta} + \Omega_b\right)\right) = -\frac{\partial \phi}{\partial \theta} \end{cases}. \tag{2}$$

We assume a weak bar with

$$\phi\left(R, \theta\right) = \phi_0\left(R\right) + \phi_1\left(R, \theta\right), \qquad \left|\frac{\phi_1}{\phi_0}\right| \ll 1, \tag{3}$$

where ϕ_0 represents the cylindrical symmetry, while ϕ_1 the perturbation. We then split the motion in two parts

$$\begin{cases} R\left(t\right) = R_0 + R_1\left(t\right) \\ \theta\left(t\right) = \theta_0\left(t\right) + \theta_1\left(t\right) \end{cases} \tag{4}$$

with R_0 the radius of the guiding centre (circular orbit). The goal is then to develop the equation of motions at the first order and interpret both the zero and first order terms. To do so, we first need to Taylor expand the potential:

$$\phi\left(R, \theta\right) \cong \phi_0\left(R_0\right) + \phi_1\left(R_0, \theta\right) + \left.\frac{\partial \phi_0}{\partial R}\right|_{R_0}\left(R - R_0\right) + \left.\frac{\partial \phi_1}{\partial R}\right|_{R_0}\left(R - R_0\right) \tag{5}$$

$$+ \left.\frac{1}{2}\frac{\partial^2 \phi_0}{\partial R^2}\right|_{R_0}\left(R - R_0\right)^2 + \left.\frac{1}{2}\frac{\partial^2 \phi_1}{\partial R^2}\right|_{R_0}\left(R - R_0\right)^2, \tag{6}$$

Then, differentiating the potential with respect to R and θ, we get:

$$\begin{cases} \frac{\partial \phi}{\partial R} \cong \left.\frac{\partial \phi_0}{\partial R}\right|_{R_0} + \left.\frac{\partial \phi_1}{\partial R}\right|_{R_0} + \left.\frac{\partial^2 \phi_0}{\partial R^2}\right|_{R_0}\left(R - R_0\right) \\ \frac{\partial \phi}{\partial \theta} \cong \left.\frac{\partial \phi_1}{\partial \theta}\right|_{R_0} \end{cases}. \tag{7}$$

Note, we drop the $\left.\frac{\partial^2 \phi_1}{\partial R^2}\right|_{R_0}\left(R - R_0\right)$ term, as $\phi_1 \ll \phi_0$.

Now, the goal is to introduce Eq. 3 and Eq. 7 in the equation of motion (1) and discuss the terms of different orders.

Zero order terms :

1. Radial equation

$$\ddot{R} = R\left(\dot{\theta} + \Omega_b\right)^2 - \frac{\partial \phi}{\partial R} \rightarrow R_0\left(\dot{\theta}_0 + \Omega_b\right)^2 = \left.\frac{\partial \phi_0}{\partial R}\right|_{R_0}; \tag{8}$$

2. Azimuthal equation

$$\frac{d}{dt}\left(R^2\left(\dot{\theta} + \Omega_b\right)\right) = -\frac{\partial \phi}{\partial \theta} \rightarrow \dot{\theta}_0 = const. \tag{9}$$

Interpretation: in absence of perturbation, the circular frequency at the radius R_0 write:

$$\Omega^2\left(R_0\right) = \frac{1}{R_0}\left.\frac{\partial \phi_0}{\partial R}\right|_{R_0}, \tag{10}$$

thus, Eq. 8 leads to

$$\dot{\theta}_0 + \Omega_b = \Omega\left(R_0\right) = \Omega_0, \tag{11}$$

and the angular frequency in the rotating rest frame is then

$$\theta_0\left(t\right) = \left(\Omega_0 - \Omega_b\right)t. \tag{12}$$

First order terms :

1. Radial equation

$$\ddot{R} = R\left(\dot{\theta} + \Omega_b\right)^2 - \frac{\partial \phi}{\partial R} \rightarrow \ddot{R}_1 + R_1\left.\left(\frac{\partial^2 \phi_0}{\partial R^2} - \Omega_b^2\right)\right|_{R_0} - 2R_0\dot{\theta}_1\Omega_0 = -\left.\frac{\partial \phi_1}{\partial R}\right|_{R_0}; \tag{13}$$

2. Azimuthal equation

$$\frac{d}{dt}\left(R^2\left(\dot{\theta} + \Omega_b\right)\right) = -\frac{\partial \phi}{\partial \theta} \rightarrow \ddot{\theta}_1 + 2\Omega_0\frac{\dot{R}_1}{R_0} = -\frac{1}{R_0^2}\left.\frac{\partial \phi_1}{\partial \theta}\right|_{R_0}. \tag{14}$$

To move forwards, we have go guess some specific potential. We assume the perturbation of the type:

$$\phi_1\left(R, \theta\right) = \phi_b\left(R\right)\cos\left(m\theta\right). \tag{15}$$

$m = 2$ corresponds to a bar. Note also that any other perturbation can be obtained by summing over m. Assuming $\theta_1 \ll \theta_0$, the gradients of Eq. 7 can now be written as:

$$\begin{cases} \frac{\partial \phi_1}{\partial R} = \frac{\phi_b}{\partial R}\cos\left(m\theta\right) \approx \frac{\partial \phi_b}{\partial R}\cos\left(m\theta_0\right) = \frac{\partial \phi_b}{\partial R}\cos\left(m\left(\Omega_0 - \Omega_b\right)t\right) \\ \frac{\partial \phi_1}{\partial \theta} = -\phi_b\left(R\right)\sin\left(m\theta\right)m \approx -\phi_b\left(R\right)m\sin\left(m\left(\Omega_0 - \Omega_b\right)t\right) \end{cases}. \tag{16}$$

Introducing those gradients in Eq 13 and 2, we get:

$$\begin{cases} \ddot{R}_1 + R_1\left.\left(\frac{\partial^2 \phi_0}{\partial R^2} - \Omega^2\right)\right|_{R_0} - 2R_0\dot{\theta}_1\Omega_0 = -\left.\frac{\partial \phi_b}{\partial R}\right|_{R_0}\cos\left(m\left(\Omega_0 - \Omega_b\right)t\right) \\ \ddot{\theta}_1 + 2\Omega_0\frac{\dot{R}_1}{R_0} = \frac{m\phi_b(R_0)}{R_0^2}\sin\left(m\left(\Omega_0 - \Omega_b\right)t\right) \end{cases}. \tag{17}$$

At this stage, it is possible to integrate $\ddot{\theta}_1$ over time:

$$\dot{\theta}_1 = -2\Omega_0 \frac{R_1}{R_0} - \frac{\phi_b(R_0)}{R_0^2(\Omega_0 - \Omega_b)} \cos(m(\Omega_0 - \Omega_b)t) + const, \tag{18}$$

and replacing it in the equation for $\ddot{R}_1$ we find

$$\ddot{R}_1 + \kappa_0^2 R_1 = -\left[\frac{\partial \phi_b}{\partial R} + \frac{2\Omega_0 \phi_b}{R(\Omega_0 - \Omega_b)}\right]_{R_0} \cos(m(\Omega_0 - \Omega_b)t) + const. \tag{19}$$

Note that we have used the radial epicycle frequency:

$$\kappa_0^2 = \left(\frac{\partial^2 \phi}{\partial R^2} + 3\Omega^2\right)\Big|_{R_0}. \tag{20}$$

The general solution is an harmonic oscillator of frequency κ_0 driven at frequency $m(\Omega_0 - \Omega_b)$. Using Eq. 12 we find

$$R_1(\theta_0) = C_1 \cos\left(\frac{\chi_0 \theta_0}{\Omega_0 - \Omega_b} + \alpha\right) - \left[\frac{\partial \phi_b}{\partial R} + \frac{2\Omega_0 \phi_b}{R(\Omega_0 - \Omega_b)}\right]_{R_0} \frac{\cos(m\theta_0)}{\chi_0^2 - m^2(\Omega_0 - \Omega_b)^2}, \tag{21}$$

with C_1 and α arbitrary constants.

Problem 2:

We start from the Hamiltonian a 1-D harmonic oscillator, assuming a frequency ω:

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$$

The equation of motion (Hamilton equations) are:

$$\begin{cases} \dot{x} = \dot{x} \\ \ddot{x} = -\omega^2 x \end{cases}$$

with the general solution:

$$\begin{cases} x(t) = A\cos(\omega t + \alpha) + B\sin(\omega t + \alpha) \\ \dot{x}(t) = -A\omega\sin(\omega t + \alpha) + B\omega\cos(\omega t + \alpha) \end{cases}$$

Taking the square, we get:

$$\begin{cases} x^2(t) = A^2\cos^2(\omega t + \alpha) + B^2\sin^2(\omega t + \alpha) + 2AB\cos^2(\omega t + \alpha)\sin^2(\omega t + \alpha) \\ \dot{x}^2(t) = A^2\omega^2\sin^2(\omega t + \alpha) + B^2\omega^2\cos^2(\omega t + \alpha) - 2AB\omega^2\cos^2(\omega t + \alpha)\sin^2(\omega t + \alpha) \end{cases}$$

and thus:

$$x(t)^2 + \frac{\dot{x}^2(t)}{w^2} = A^2 + B^2$$

which is the equation of an ellispe of ellipticity w (if $w < 1$) or $1/w$ (if $w > 1$). Assuming $\alpha = 0$ we have:

$$\begin{cases} A = x_0 \\ B = v_0/\omega \end{cases}$$

and the evolution is equivalent to multiplying the initial position and velocity by a matrix (time operator):

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}$$

The geometrical interpretation of the matrix is to apply a rotation with some deformation (if $w = 1$, this is a pure rotation). But as the determinant of the matrix is 1, the area is conserved and thus the density of the phase space too.