Solutions to Homework 9
CS-526 Learning Theory

Problem 1

B is true. If $w_i(\epsilon) \neq 0$ for all i then the three arrays have rank K and there are K terms in the tensor decomposition. Therefore, by Jennrich's theorem, the decomposition is unique and the rank of the tensor $T(\epsilon)$ is K.

A is not true if there exists some (i, ϵ) such that $w_i(\epsilon)$ is zero.

C is not true because all the functions w_i are continuous. Therefore, $\lim_{\epsilon \to 0} T(\epsilon) = T(0)$ and, by Jennrich's theorem, the rank is at most K (the rank is K if $\forall i : w_i(0) \neq 0$).

D is not true because if $w_i(\epsilon) \neq 0$ for all i and $\epsilon \in [0, 1]$ then the rank is K.

Problem 3

The first identity simply follows from the definitions:

$$(\mathbf{c} \otimes_{\text{Kro}} \mathbf{b})^T = \begin{bmatrix} c_1\mathbf{b}^T & c_2\mathbf{b}^T & \cdots & c_J\mathbf{b}^T \end{bmatrix} = \mathbf{c}^T \otimes_{\text{Kro}} \mathbf{b}^T \ .$$

For the second identity on the inner product between the two column vectors $\mathbf{e} \otimes_{\text{Kro}} \mathbf{d}$ and $\mathbf{c} \otimes_{\text{Kro}} \mathbf{b}$, we simply have:

$$(\mathbf{e} \otimes_{\text{Kro}} \mathbf{d})^T (\mathbf{c} \otimes_{\text{Kro}} \mathbf{b}) = \begin{bmatrix} e_1\mathbf{d}^T & e_2\mathbf{d}^T & \cdots & e_J\mathbf{d}^T \end{bmatrix} \begin{bmatrix} c_1\mathbf{b} \\ c_2\mathbf{b} \\ \vdots \\ c_J\mathbf{b} \end{bmatrix} = \sum_{j=1}^{J} e_j c_j \mathbf{d}^T\mathbf{b} = (\mathbf{e}^T\mathbf{c})(\mathbf{d}^T\mathbf{b}) \ .$$

Finally, the product of the $R \times IJ$ matrix $(E \otimes_{\text{Khr}} D)^T$ and the $IJ \times R$ matrix $(C \otimes_{\text{Khr}} B)$ is the $R \times R$ matrix whose entries are $\forall (i, j) \in \{1, \ldots, R\}^2$:

$$\begin{aligned}
\big[(E \otimes_{\text{Khr}} D)^T (C \otimes_{\text{Khr}} B)\big]_{ij} &= \sum_{k=1}^{IJ} \big[E \otimes_{\text{Khr}} D\big]_{ki} \big[C \otimes_{\text{Khr}} B\big]_{kj} \\
&= (\mathbf{e}_i \otimes_{\text{Kro}} \mathbf{d}_i)(\mathbf{c}_j \otimes_{\text{Kro}} \mathbf{b}_j) \\
&= (\mathbf{e}_i^T \mathbf{c}_j)(\mathbf{d}_i^T \mathbf{b}_j) \\
&= [E^T C]_{ij} [D^T B]_{ij} \\
&= [(E^T C) \circ (D^T B)]_{ij} \ .
\end{aligned}$$

The third equality follows from the identity on the inner product of two Kronecker products. Hence $(E \otimes_{\text{Khr}} D)^T (C \otimes_{\text{Khr}} B) = (E^T C) \circ (D^T B)$.