Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

The Jeans equations are obtained from the Boltzmann equations, by computing moments of various orders.
A- Direct integration on velocities (moment of order 0)
B- Integration on the velocities after multiplying by one component of the velocity (moment of order 1)
Here are a few properties to keep in mind :

1) $f \to 0$ when $|v_i| \to \infty$ 2) $m \int f\, d^3\mathbf{v} = \rho$ 3) $m \int v_i\, f\, d^3\mathbf{v} = \rho \overline{v_i}$

4) $\int v_i\, v_j f\, d^3\mathbf{v} = \rho \overline{v_i v_j}$ 5) $\overline{v_i}\,\overline{v_j} + {\sigma_{ij}}^2 = \overline{v_i v_j}$

where we set $m = 1$.

A - moment 0:

$$\frac{\partial \nu}{\partial t} + \sum_i \frac{\partial}{\partial x_i} \left(\nu \overline{v_i}\right) = 0$$

in vectorial notation:

$$\frac{\partial \nu}{\partial t} + \nabla \cdot \left(\nu\, \overline{\mathbf{v}}\right) = 0$$

In spherical coordinates, the divergence of a vector reads :

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, F_\theta) + \frac{1}{r\sin\theta} \frac{\partial F_\phi}{\partial \phi}$$

consequently, the equation becomes :

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial r}\left(\nu \overline{v_r}\right) + \frac{2}{r}\,\nu \overline{v_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\nu \overline{v_\theta}\right) + \frac{\cot\theta}{r} \nu \overline{v_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\nu \overline{v_\phi}\right) = 0$$

The systems with a spherical symmetry have negligible meridional motions, hence $\overline{v_\theta} = 0$. Furthermore, a possible rotation of the system is done at an azimuthal symmetry, i.e. $\partial \overline{v_\phi}/\partial \phi = 0$. (In short, there can be no angular dependencies in a spherically symmetric system, hence $\partial/\partial\theta = 0$, $\partial/\partial\phi = 0$)

Thus, we get for the moment 0

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho \overline{v_r}) = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left(\rho \overline{v_r}\right) + \frac{2}{r}\,\rho \overline{v_r} = 0$$

B - First moment In vectorial notation

$$\frac{\partial \overline{\mathbf{v}}}{\partial t} + (\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} = -\nabla\Phi - \frac{1}{\rho}\,\nabla \cdot (\rho\boldsymbol{\sigma}^{\mathbf{2}})$$

Transformation to spherical coordinates is risky (because of the divergence of tensor), so it is better to start directly from the collisionless Boltzmann equation expressed in spherical coordinates.

$$\frac{\partial f}{\partial t} + v_r\,\frac{\partial f}{\partial r} + \frac{v_\theta}{r}\,\frac{\partial f}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\,\frac{\partial f}{\partial \phi} + \left(\frac{v_\theta^2 + v_\phi^2}{r} - \frac{\partial \Phi}{\partial r}\right)\,\frac{\partial f}{\partial v_r}$$

$$+\frac{1}{r}\left(v_\phi^2\cot\theta - v_r v_\theta\right)\,\frac{\partial f}{\partial v_\theta} - \frac{1}{r}\left[v_\phi\left(v_r + v_\theta\cot\theta\right)\right]\,\frac{\partial f}{\partial v_\phi} = 0$$

We compute the radial Jeans equation by multiplying the collisionless Boltzmann equation by v_r and integrating on velocities

$$\int v_r^2 \frac{\partial f}{\partial r}\,d^3\mathbf{v} = \frac{\partial}{\partial r}\int f\,v_r^2\,d^3\mathbf{v} = \frac{\partial}{\partial r}\left(\rho\,\overline{v_r^2}\right)$$

$$\int \frac{v_r\,v_\theta}{r}\,\frac{\partial f}{\partial \theta}\,d^3\mathbf{v} = \frac{1}{r}\,\frac{\partial}{\partial \theta}\,\int f\,v_r\,v_\theta\,d^3\mathbf{v} = \frac{1}{r}\,\frac{\partial}{\partial \theta}\,(\rho\,\overline{v_r v_\theta}) = 0$$

$$\int \frac{v_r\,v_\phi}{r\sin\theta}\,\frac{\partial f}{\partial \phi}\,d^3\mathbf{v} = \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \phi}\,\int f\,v_r\,v_\phi\,d^3\mathbf{v} = \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \phi}\,(\rho\,\overline{v_r v_\phi}) = 0$$

where the null values in the last two equations comes from the assumption of spherical symmetry,

$$\int \frac{v_r\,v_\theta^2}{r}\,\frac{\partial f}{\partial v_r}\,d^3\mathbf{v} = \frac{1}{r}\int dv_\phi \int v_\theta^2\,dv_\theta \int v_r \frac{\partial f}{\partial v_r}\,dv_r = -\frac{1}{r}\int f\,v_\theta^2\,d^3\mathbf{v} = -\rho\frac{\overline{v_\theta^2}}{r}$$

where the integral on v_r was integrated by parts, and similarly,

$$\int \frac{v_r\,v_\phi^2}{r}\frac{\partial f}{\partial v_r}\,d^3\mathbf{v} = \frac{1}{r}\int dv_\theta \int v_\phi^2\,dv_\phi \int v_r \frac{\partial f}{\partial v_r}\,dv_r = -\frac{1}{r}\int f\,v_\phi^2\,d^3\mathbf{v} = -\rho\frac{\overline{v_\phi^2}}{r}$$

$$\int \frac{\partial \Phi}{\partial r}\,v_r\,\frac{\partial f}{\partial v_r}\,d^3\mathbf{v} = \frac{\partial \Phi}{\partial r}\int dv_\phi \int dv_\theta \int v_r \frac{\partial f}{\partial v_r}\,dv_r = -\frac{\partial \Phi}{\partial r}\,\int f\,d^3\mathbf{v} = -\rho\,\frac{\partial \Phi}{\partial r}$$

still with the same integration by parts,

$$\int v_r v_\phi^2\,\frac{\cot\theta}{r}\frac{\partial f}{\partial v_\theta}\,d^3\mathbf{v} = \frac{\cot\theta}{r}\,\int v_r\,dv_r\,\int v_\phi^2\,dv_\phi\,\int \frac{\partial f}{\partial v_\theta}\,dv_\theta = 0$$

after integration by parts of the integral on v_θ,

$$\int \frac{v_r^2 v_\theta}{r}\,\frac{\partial f}{\partial v_\theta}\,d^3\mathbf{v} = \frac{1}{r}\int v_r^2\,dv_r \int dv_\phi \int v_\theta \frac{\partial f}{\partial v_\theta}\,dv_\theta = -\frac{1}{r}\int f\,v_r^2\,d^3\mathbf{v} = -\frac{\rho\overline{v_r^2}}{r}$$

and similarly,

$$\int \frac{v_r^2 v_\phi}{r}\,\frac{\partial f}{\partial v_\phi}\,d^3\mathbf{v} = \frac{1}{r}\int v_r^2\,dv_r \int dv_\theta \int v_\phi \frac{\partial f}{\partial v_\phi}\,dv_\phi = -\frac{1}{r}\int f\,v_r^2\,d^3\mathbf{v} = -\frac{\rho\overline{v_r^2}}{r}$$

and finally,

$$\begin{aligned}\int \frac{v_r v_\theta v_\phi\,\cot\theta}{r}\,\frac{\partial f}{\partial v_\phi}\,d^3\mathbf{v} &= \frac{\cot\theta}{r}\int v_r dv_r \int v_\theta\,dv_\theta \int v_\phi \frac{\partial f}{\partial v_\phi}\,dv_\phi \\ &= -\frac{\cot\theta}{r}\int v_r v_\theta f d^3\mathbf{v} = -\frac{\rho\overline{v_r v_\theta}\cot\theta}{r}\end{aligned}$$

where we have again performed an integration by parts for the integral on v_ϕ. Since we're in a spherically symmetric case, we may choose any fixed θ, and we choose θ such that $\cot\theta = 0$.
Putting everything together finally results in the general Jeans equation for spherical symmetry:

$$\frac{\partial\left(\rho\overline{v_r}\right)}{\partial t} + \frac{\partial\left(\rho\overline{v_r^2}\right)}{\partial r} + \frac{\rho}{r}\left[2\,\overline{v_r^2} - \left(\overline{v_\theta^2} + \overline{v_\phi^2}\right)\right] = -\rho\,\frac{\partial\Phi}{\partial r}$$

One can introduce the velocity dispersion : $\overline{v_i^2} = \sigma_i^2 + \overline{v_i}^2$
Isotropic systems: $\overline{v_\phi} = \overline{v_\theta} = \overline{v_r}$
For a stationary system with isotropic velocities, the Jeans equation reduces to :

$$\frac{d\left(\rho\sigma_r^2\right)}{dr} = -\rho\,\frac{d\Phi}{dr}$$

The potential Φ in the Jeans equation is always the gravitational potential representing the total mass of the system. ρ may be a mass density, a number density or even a luminosity density.

Problem 2:

Plummer:

$$\begin{aligned}\rho &= \frac{3\,M}{4\pi\,a^3}\left[1 + \left(\frac{r}{a}\right)^2\right]^{-5/2} \\ \Phi &= -\frac{GM}{\sqrt{r^2+a^2}} \\ \frac{d\Phi}{dr} &= GMr(r^2+a^2)^{-3/2}\end{aligned}$$

Introducing these expressions into the last equation of Problem 2, we get

$$\begin{aligned}\frac{d\left(\rho\sigma_r^2\right)}{dr} &= -\frac{3\,M}{4\pi\,a^3}\left[1+\left(\frac{r}{a}\right)^2\right]^{-5/2}\cdot GMr\,\left(r^2+a^2\right)^{-3/2}\\ &= -\frac{3GM^2a^2}{4\pi}\frac{r}{\left(a^2+r^2\right)^{5/2}\left(a^2+r^2\right)^{3/2}} = -\frac{3GM^2a^2}{4\pi}\frac{r}{\left(a^2+r^2\right)^4}\end{aligned}$$

By integration, taking into account that $\rho\sigma_r^2$ must tend to zero when M tends to zero, one obtains

$$\rho\sigma_r^2 = \frac{GM^2a^2}{8\pi\left(r^2+a^2\right)^3}$$

Finally,

$$\sigma_r^2 = \frac{GM}{6\sqrt{r^2+a^2}}$$

Problem 3:

From collisonless Boltzmann equation in cylindrical coordinates in term of velocities write:

$$\frac{\partial f}{\partial t}+v_R\frac{\partial f}{\partial R}+\frac{v_\phi}{R}\frac{\partial f}{\partial\phi}+v_z\frac{\partial f}{\partial z}+\left[\frac{v_\phi^2}{R}-\frac{\partial\Phi}{\partial R}\right]\frac{\partial f}{\partial v_R}-\frac{v_Rv_\phi}{R}\frac{\partial f}{\partial v_\phi}-\frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z}=0 \quad (1)$$

Assuming a steady state ($\frac{\partial f}{\partial t}=0$) and an azimuthal symmetry ($\frac{\partial\Phi}{\partial\phi}=0$, $\frac{\partial f}{\partial\phi}=0$), we get:

$$v_R\frac{\partial f}{\partial R}+v_z\frac{\partial f}{\partial z}+\left[\frac{v_\phi^2}{R}-\frac{\partial\Phi}{\partial R}\right]\frac{\partial f}{\partial v_R}-\frac{v_Rv_\phi}{R}\frac{\partial f}{\partial v_\phi}-\frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z}=0 \quad (2)$$

The first moment in v_R writes:

$$\int dv_Rdv_\phi dv_z v_R\left[v_R\frac{\partial f}{\partial R}+v_z\frac{\partial f}{\partial z}+\left[\frac{v_\phi^2}{R}-\frac{\partial\Phi}{\partial R}\right]\frac{\partial f}{\partial v_R}-\frac{v_Rv_\phi}{R}\frac{\partial f}{\partial v_\phi}-\frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z}\right]=0 \quad (3)$$

Using the rules given in Problem 1, we can write:

$$\int dv_Rdv_\phi dv_z v_Rv_R\frac{\partial f}{\partial R} = \frac{\partial}{\partial R}\left(\nu\overline{v_R^2}\right) \quad (4)$$

$$\int dv_Rdv_\phi dv_z v_Rv_z\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\nu\overline{v_Rv_z}\right) \quad (5)$$

$$\int dv_Rdv_\phi dv_z v_R\frac{v_\phi^2}{R}\frac{\partial f}{\partial v_R} = -\frac{\nu}{R}\overline{v_\phi^2} \quad (6)$$

$$-\int dv_Rdv_\phi dv_z v_R\frac{\partial\Phi}{\partial R}\frac{\partial f}{\partial v_R} = \nu\frac{\partial\Phi}{\partial R} \quad (7)$$

$$-\int dv_Rdv_\phi dv_z v_R\frac{v_Rv_\phi}{R}\frac{\partial f}{\partial v_\phi} = \frac{\nu}{R}\overline{v_R^2} \quad (8)$$

$$-\int dv_R dv_\phi dv_z v_R \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0 \tag{9}$$

Putting all together, we get:

$$\frac{\partial}{\partial R}\left(\nu\overline{v_R^2}\right) + \frac{\partial}{\partial z}\left(\nu\overline{v_R v_z}\right) + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R}\right) = 0 \tag{10}$$

The the first two other moment are obtained by successively multiplying by v_ϕ and v_z and integrating over the velocities. Using the same mathematical tricks, we obtain:

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\nu\overline{v_R v_z}\right) + \frac{\partial}{\partial z}\left(\nu\overline{v_z^2}\right) + \nu\frac{\partial \Phi}{\partial z} = 0, \tag{11}$$

and:

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\nu\overline{v_R v_\phi}\right) + \frac{\partial}{\partial z}\left(\nu\overline{v_z v_\phi}\right) = 0. \tag{12}$$