Quantum computation : lecture 13

Reminder : classical codes

$$\begin{array}{ccc} 0 & \longrightarrow & 000 \\ 1 & & 111 \end{array}$$

$d=3$ in this repetition code

info of length k — codewords of length n

These codewords go through a channel:

codeword x —— $\boxed{\text{channel}}$ —— y output

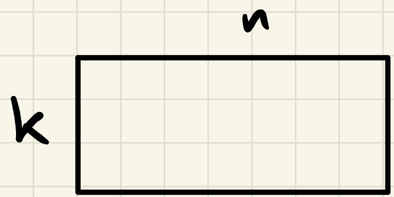
Simple error model: (bit-flip) $y = x + e$, $e = 010$ eg.

bit flip

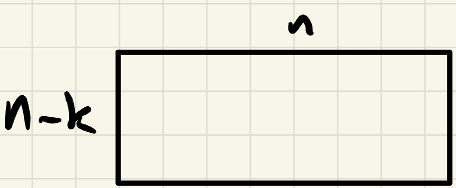
This repetition code can correct up to $\lfloor \frac{d-1}{2} \rfloor = 1$ error

If probability(bit flip) $= p < \frac{1}{2}$, then the majority rule outputs the most likely info bit 0 or 1.

<u>General linear (n, k, d) code:</u>



= generator matrix G



= parity-check matrix H

$$G \cdot H^T = 0$$

$$\mathcal{C} = \{ x \in \mathbb{F}_2^n : x = u \cdot G, \, u \in \mathbb{F}_2^k \} = \{ x \in \mathbb{F}_2^n : H x^T = 0 \}$$

Ex: Hamming code $n=7$, $k=4$, $d=3$

$$H = \overbrace{\begin{pmatrix} 0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1 \end{pmatrix}}^{n=7} \begin{matrix} n-k \\ =3 \end{matrix}$$

all non-zero columns of length 3

$$G = \overbrace{\begin{pmatrix} 0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1 \\ 1&1&1&1&1&1&1 \end{pmatrix}}^{n=7} k=4$$

$2^4 = 16$ codewords

$d=3$ here: no column in H is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

& no two columns in H are linearly dep.

$\Rightarrow d \geq 3$ (and $d=3$ because 1110000 is a codeword)

This code can therefore correct 1 error

<u>Syndrom decoding</u>:

Codeword x $\longrightarrow$ output $y = x + e$

$$H y^T = H(x^T + e^T) = \underbrace{H x^T}_{=0} + H e^T = H e^T$$

assume $e = 0000100$ $\Rightarrow$ $He^T =$ column which is the binary rep. of i (ex: $\begin{pmatrix}1\\0\\1\end{pmatrix}$)

$\uparrow$ position i (ex: $i = 5$)

Quantum error correction

<u>repetition code</u>: $\begin{cases} |0\rangle \longrightarrow |000\rangle \\ |1\rangle \longrightarrow |111\rangle \end{cases}$

⚠ this is not cloning!

(cloning would be $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |\varphi\rangle \otimes |\varphi\rangle \otimes |\varphi\rangle$)

Here we do:

$$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |\psi\rangle = \alpha|000\rangle + \beta|111\rangle$$

"codewords"

Error models : $\overset{X}{\text{bit-flip}}$ or $\overset{Z}{\text{phase-flip}}$

(NB: we could think about other types of errors, like variations of α & β ... later!)

bit-flip: (in position 1, e.g.)

$$|\psi\rangle \;—\boxed{X_1}—\; |\psi'\rangle = \overbrace{(X \otimes I \otimes I)}^{= X_1} |\psi\rangle$$

$$= \alpha|000\rangle + \beta|111\rangle \qquad\qquad = \alpha|100\rangle + \beta|011\rangle$$

Measurements (→ "syndroms")

Observables : $\underbrace{Z_1 Z_2}_{= Z \otimes Z \otimes I}$ & $\underbrace{Z_2 Z_3}_{= I \otimes Z \otimes Z}$

possible eigenvalues $+1$ & -1

NB: remember the parity-check matrix of the classical repetition code :

$$H = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{l} Z_1 Z_2 \\ Z_2 Z_3 \end{array}$$

Assume no error happened: $|\psi'\rangle = \alpha|000\rangle + \beta|111\rangle$

$$Z_1 Z_2 |\psi'\rangle = \alpha|000\rangle + \underbrace{(-1)\cdot(-1)}_{=+1}\beta|111\rangle = (+1)\cdot|\psi'\rangle$$

$$Z_2 Z_3 |\psi'\rangle = (+1)\cdot|\psi'\rangle \quad \text{(likewise)}$$

Assume bit 1 was flipped: $|\psi'\rangle = \alpha|100\rangle + \beta|011\rangle$

$$Z_1 Z_2 |\psi'\rangle = -\alpha|100\rangle - \beta|011\rangle = (-1)|\psi'\rangle$$

$$Z_2 Z_3 |\psi'\rangle = +\alpha|100\rangle + \beta|011\rangle = (+1)|\psi'\rangle$$

In summary: measurement of $\underbrace{Z_1 Z_2 \;\&\; Z_2 Z_3}_{= \text{“stabilizers”}}$ gives:

$(+1, +1) \longleftrightarrow$ no bit flip

$(-1, +1) \longleftrightarrow$ bit-flip in position 1

$(-1, -1) \longleftrightarrow$ bit-flip in position 2

$(+1, -1) \longleftrightarrow$ bit-flip in position 3

$\uparrow$

our new syndroms

(observe that $|\psi'\rangle$ an eigenvector of Z_1, Z_2, Z_3 in all cases: no state perturbation!)

<u>Error detection</u> is done; now we need to do <u>error correction</u>:

- if $(+1,+1) \longrightarrow$ do nothing
- if $(-1,+1) \longrightarrow$ apply $X_1 \longrightarrow \left\{ \begin{array}{l} \text{back to} \\ \text{state } |\psi\rangle = \alpha|000\rangle + \beta|111\rangle \end{array} \right.$

Note X_1 need to be applied <u>after</u> the Z's to $|\psi'\rangle$, which is <u>not</u> an eigenvector of X_1

(and the two other cases are similar)

In summary:

- $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle \xrightarrow[\text{(bit-flip)}]{\text{channel}} |\psi'\rangle = X_1|\psi\rangle$
$$= \alpha|100\rangle + \beta|011\rangle$$

- error detection: $Z_1 Z_2 |\psi'\rangle = (-1)|\psi'\rangle$

  and $Z_2 Z_3 |\psi'\rangle = (+1)|\psi'\rangle$, $|\psi'\rangle$ unchanged

- error correction: $X_1|\psi'\rangle = X_1 \cdot X_1 |\psi\rangle = |\psi\rangle$

Other possible error : phase-flip $\begin{cases} Z|0\rangle = |0\rangle \\ Z|1\rangle = (-1)|1\rangle \end{cases}$

$\Rightarrow$ new code $\begin{cases} |0\rangle \longrightarrow |+++\rangle \\ |1\rangle \longrightarrow |---\rangle \end{cases}$ $\quad |\psi\rangle = \alpha|+++\rangle + \beta|---\rangle$

$Z|+\rangle = |-\rangle$, $Z|-\rangle = |+\rangle$ $\Rightarrow$ same as before!

phase-flip: $Z_1|\psi\rangle = \alpha|-++\rangle + \beta|+--\rangle$

error detection: observables X_1X_2 & X_2X_3

error correction: if $(-1,+1)$, apply Z_1 to

recover the original state!

How to handle bit & phase-flips together?

$\longrightarrow$ <u>Shor's code</u> (= concatenation of the two previous codes)

<u>Step 1</u>: (good for phase-flips)

$$|0\rangle \longrightarrow |+++\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|1\rangle \longrightarrow |---\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Step 2 : (good for bit-flips)

$$|0\rangle \longrightarrow \frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle + |111\rangle}{\sqrt{2}} = |0\rangle_{\text{Shor}}$$

$$|1\rangle \longrightarrow \frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}} \otimes \frac{|000\rangle - |111\rangle}{\sqrt{2}} = |1\rangle_{\text{Shor}}$$

Observe $k=1$ & $n=9$ here.

Claim: This code protects against a bit-flip and/or a phase-flip.

- Initial state: $|\psi\rangle = \alpha|0\rangle_{shor} + \beta|1\rangle_{shor}$

$\Rightarrow$ output state $|\psi'\rangle$ (bit-flip and/or phase-flip)

- Stabilizers : $Z_1 Z_2,\ Z_2 Z_3,\ Z_4 Z_5,\ Z_5 Z_6,\ Z_7 Z_8,\ Z_8 Z_9$

These measuments will not perturb the state and will provide info on the bit-flip.

- Next stabilizers : $X_1 X_2 X_3 X_4 X_5 X_6,\ X_4 X_5 X_6 X_7 X_8 X_9$

These will provide info on the phase-flip.

<u>Fact</u>: All these operators commute and $|\psi'\rangle$ is an eigenvector of all of them

<u>Error correction</u>: apply the correct Z or X !

<u>Ex 1</u>: bit-flip error on bit 3

In this case, only $Z_2 Z_3$ has eigenvalue -1

$\Rightarrow$ one applies X_3 to correct the error

Ex 2: phase-flip error on bit 5

In this case, both $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$ have eigenvalue -1; one does not know where the phase-flip occured among bits 4, 5 or 6, but one can still correct the error by applying $Z_4Z_5Z_6$!

Ex 3: bit-flip and phase-flip error on bit 4

In this case, $Z_4 Z_5$, $X_1 X_2 X_3 X_4 X_5 X_6$ and $X_4 X_5 X_6 X_7 X_8 X_9$ have eigenvalue -1

This bit-phase flip can be corrected by applying both X_4 & Z_4 (note that depending on the order, this might generate a global (-1) phase, as $X_4 Z_4 = -Z_4 X_4$, but the error will be corrected).

Steane's code: $k=1, \quad n=7$ (more efficient)

$$
\begin{cases}
|0\rangle \to \frac{1}{\sqrt{8}}\big(|0000000\rangle + |1010101\rangle + |0110011\rangle + |0001111\rangle \\
\qquad + |0111100\rangle + \dots\big) \\
|1\rangle \to \frac{1}{\sqrt{8}}\big(|1111111\rangle + |0101010\rangle + |1001100\rangle + |1110000\rangle \\
\qquad + |1000011\rangle + \dots\big)
\end{cases}
$$

Stabilizers:

$$X_4X_5X_6X_7 \qquad Z_4Z_5Z_6Z_7$$

$$X_2X_3X_6X_7 \qquad Z_2Z_3Z_6Z_7$$

$$X_1X_3X_5X_7 \qquad Z_1Z_3Z_5Z_7$$

Claim: All these measurements commute and $|\psi'\rangle$ = state after one bit-flip and/or phase-flip is an eigenvector.

$\Longrightarrow$ error correction: next week!